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Introduction to Real Orthogonal Polynomials

by

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN APPLIED MATHEMATICS

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ABSTRACT

The fundamental concept of orthogonality of mathematical objects occurs in a wide variety of physical and engineering disciplines. The theory of orthogonal functions, for example, is central to the development of Fourier series and wavelets, essential for signal processing. In particular, various families of classical orthogonal polynomials have traditionally been applied to fields such as electrostatics, numerical analysis, and many others.

This thesis develops the main ideas necessary for understanding the classical theory of orthogonal polynomials. Special emphasis is given to the Jacobi polynomials and to certain important subclasses and generalizations, some recently discovered. Using the theory of hypergeometric power series and their q−extensions, various structural properties and relations between these classes are systematically investigated. Recently, these classes have found significant applications in coding theory and the study of angular momentum, and hold much promise for future applications.

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I. INTRODUCTION

The abstract concept of orthogonality of functions (or other mathematical objects) is a generalization of the notion of having two or more vectors perpendicular to one another. This concept arises naturally in a wide variety of physical and engineering disciplines. For example, the theory of orthogonal functions is central to the development of Fourier series and wavelets which are essential to signal processing.

Classical Fourier series (real form) depend on the property that the trigonometric functions sine and cosine are orthogonal (on an appropriate real interval) in a formal sense that will be made precise later. As a consequence, a bounded periodic function $f(x)$ of period 2π which satisfies the Dirichlet conditions[1] may be expressed in the form

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx , \quad n = 0, 1, 2, \ldots$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx , \quad n = 1, 2, 3, \ldots$$

are the *classical Fourier coefficients.* These formulas can be modified via a change of variable to accomodate any such function of period $2L$. [Ref. 1: p. 529]

This property can be used to generate other classes of orthogonal functions - polynomials, for example - that behave in very structured and useful ways such as in generalized Fourier series. In particular, specific families of these "classical orthogonal polynomials" have traditionally been used for solving problems arising in various areas of applied mathematics, physics, and engineering, among others.

This thesis develops the main ideas necessary for understanding the classical theory of orthogonal polynomials. Special emphasis is given to the Jacobi polynomials and to certain important subclasses and generalizations. Much of the investigation will be

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[1] Dirchlet conditions: (i) In any period $f(x)$ is continuous, except possibly for a finite number of jump discontinuities, (ii) In any period $f(x)$ has only a finite number of maxima and minima.

made using the theory of hypergeometric power series and their q-extensions. The classes discussed in this thesis are but a small fraction of those identified and studied in the literature.

A. CHEBYSHEV POLYNOMIALS

The *Chebyshev polynomials of the first kind,* $T_n(x)$, arise from an elementary trigonometric consideration. As such, they satisfy various properties and identities which are easily derived directly from their definition, many of which are observable from their graphs (see below). This class of polynomials will serve as the model for some of the basic structure of more general classes.

For $n = 0,1,2, \ldots$, define

$$T_n(x) = \cos(n \arccos x), \quad -1 \le x \le 1$$

i.e., letting $x = \cos\theta$, $0 \le \theta \le \pi$,

$$T_n(\cos\theta) = \cos n\theta. \tag{1}$$

Some immediate consequences of (1) are $|T_n(x)| \le 1$ for $|x| \le 1$, with $T_n\left(\cos\frac{k\pi}{n}\right) = (-1)^k$, $0 \le k \le n$; in particular $T_n(1) = 1$ and $T_n(-1) = (-1)^n$ for all n which can be seen graphically in Figure 1.

1. Three-term Recurrence Relation/Differential Equation

From (1), we have

$$T_0(x) = 1 \text{ and } T_1(x) = x, \tag{2}$$

and by considering the identity

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b \tag{3}$$

with $a = n\theta$, $b = \theta$, we obtain

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{4}$$

Equation (4) is known as the *three-term recurrence relation* for $T_n(x)$ which together with initial conditions (2) imply that $T_n(x)$ is a polynomial of degree exactly n, called the n^{th} Chebyshev polynomial of the first kind. Note that the leading coefficient of $T_n(x)$ is 2^{n-1} for $n \ge 2$. An inductive argument applied to this recursion shows that

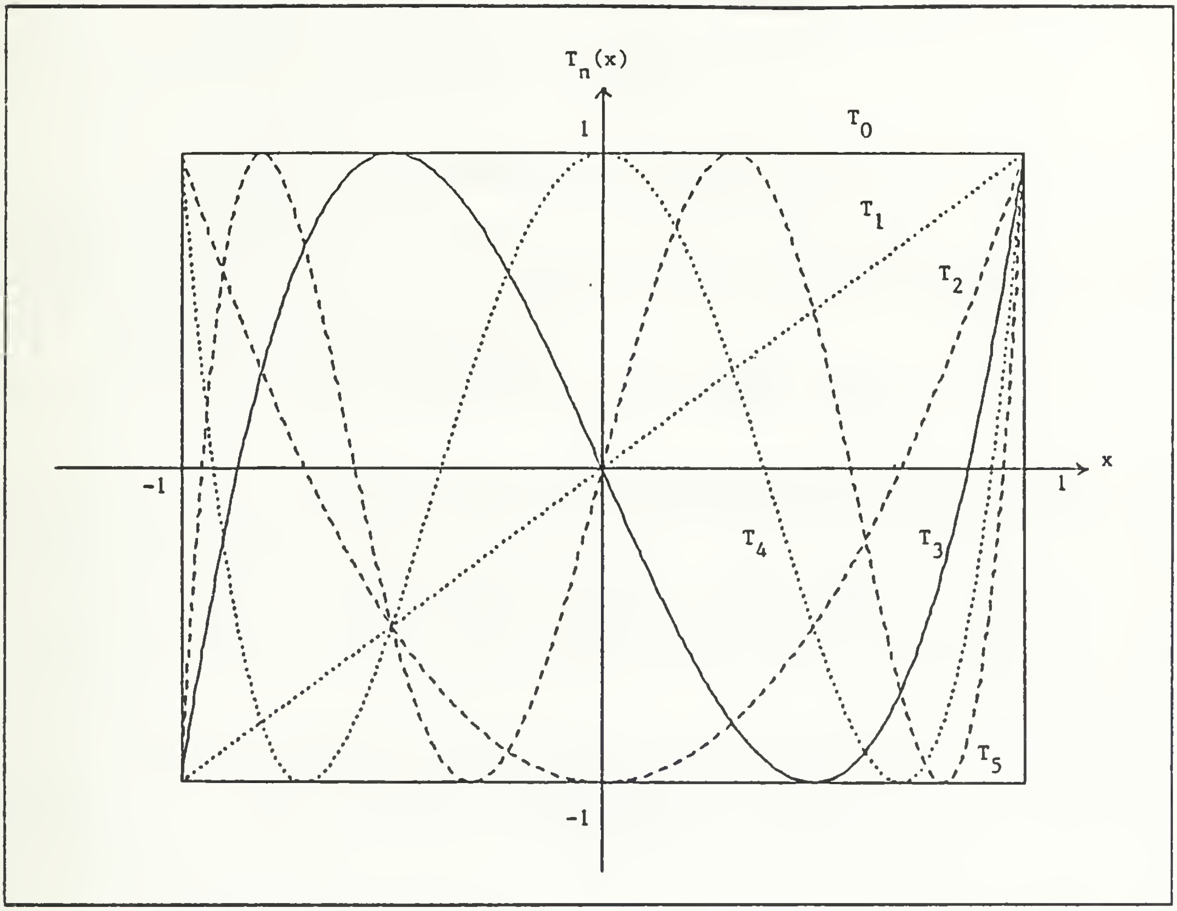

Figure 1. Chebyshev Polynomials

$T_n(x)$ is an even function if n is even, and odd if n is odd (see Figure 1). The first few are listed below:

$T_0(x) = 1\ ,\quad T_1(x) = x\ ,\quad T_2(x) = 2x^2 - 1\ ,\quad T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1\ ,\quad T_5(x) = 16x^5 - 20x^3 + 5x.$

Differentiating (1) twice with respect to θ yields the *second order differential equation* for $T_n(x)$:

(5) $$(1 - x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0.$$

2. Orthogonality of Chebyshev Polynomials

Let

$$I_{m,n} = \int_0^{\pi} \cos m\theta \cos n\theta d\theta.$$

If $m \neq n$, then using (3) with $a = m\theta,\ b = n\theta$ yields

$$I_{m,n} = \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n} \right]_0^{\pi} = 0.$$

If $m = n \neq 0$, then by using the identity $\cos^2 a = \frac{1}{2}(1 + \cos 2a)$,

$$I_{n,n} = \frac{1}{2}\left[\theta + \frac{1}{2n} \sin 2n\theta \right]_0^{\pi} = \frac{\pi}{2}.$$

If $m = n = 0$, then

$$I_{0,0} = \int_0^{\pi} d\theta = \pi.$$

Hence,

$$\int_0^{\pi} \cos m\theta \ \cos n\theta \ d\theta = h_n^{-1}\, \delta_{m,n} \tag{6}$$

where

$$h_n^{-1} = \begin{cases} \pi, & m = n = 0 \\ \pi/2, & m = n \neq 0 \end{cases}$$

and

$$\delta_{m,n} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

is the *Kronecker delta function*. Changing variables via $x = \cos\theta$, we have

(7) $$\int_{-1}^{1} T_m(x)T_n(x)\frac{1}{\sqrt{1-x^2}}\,dx = h_n^{-1}\,\delta_{m,n}.$$

This important property is formally known as the *orthogonality relation* for the Chebyshev polynomials. The reason for this terminology will become clear in the next chapter.

3. Zeros of Chebyshev Polynomials

Setting $T_n(\cos\theta) = \cos n\theta = 0$, we obtain $\theta = \theta_{k,n} = \dfrac{2k-1}{2n}\pi$, i.e., $x = x_{k,n} = \cos\theta_{k,n}$, $1 \le k \le n$.

Thus all the zeros of $T_n(x)$ are real, distinct, and may be regarded as the projections onto the interval $(-1,1)$ of the equally distributed points $\theta_{k,n}$ on the unit circle, as seen in Figure 2. Moreover, for $1 \le k \le n$, an easy algebraic check verifies that $\theta_{k,n+1} < \theta_{k,n} < \theta_{k+1,n+1}$ and therefore $x_{k,n+1} < x_{k,n} < x_{k+1,n+1}$. Hence, the zeros of $T_{n+1}(x)$ interlace with those of $T_n(x)$. This interlacing of zeros is a striking feature of the plots in Figure 1.

The zeros of Chebyshev polynomials, and of other orthogonal polynomials in general, are extremely important for applications to numerical analysis, electrostatics, and many other fields.

4. Looking Ahead

Many of the properties derived for the Chebyshev polynomials $T_n(x)$ from their trigonometric definition (1), extend to more general classes of orthogonal polynomials via a general theory, elements of which will be developed in this thesis. Some of the many properties satisfied by these classes that we will derive include:

1. Orthogonality with respect to a weight function
2. Three-term recurrence relation
3. Second order differential or difference equation
4. Hypergeometric series expression
5. Rodrigues' formula
6. Generating function.

The general approach we will take is to define these "classical orthogonal polynomials" via terminating hypergeometric power series, and from this prove (most of) the other properties. However, because of this equivalence, many authors choose to define a given class using one of these other characterizing properties.

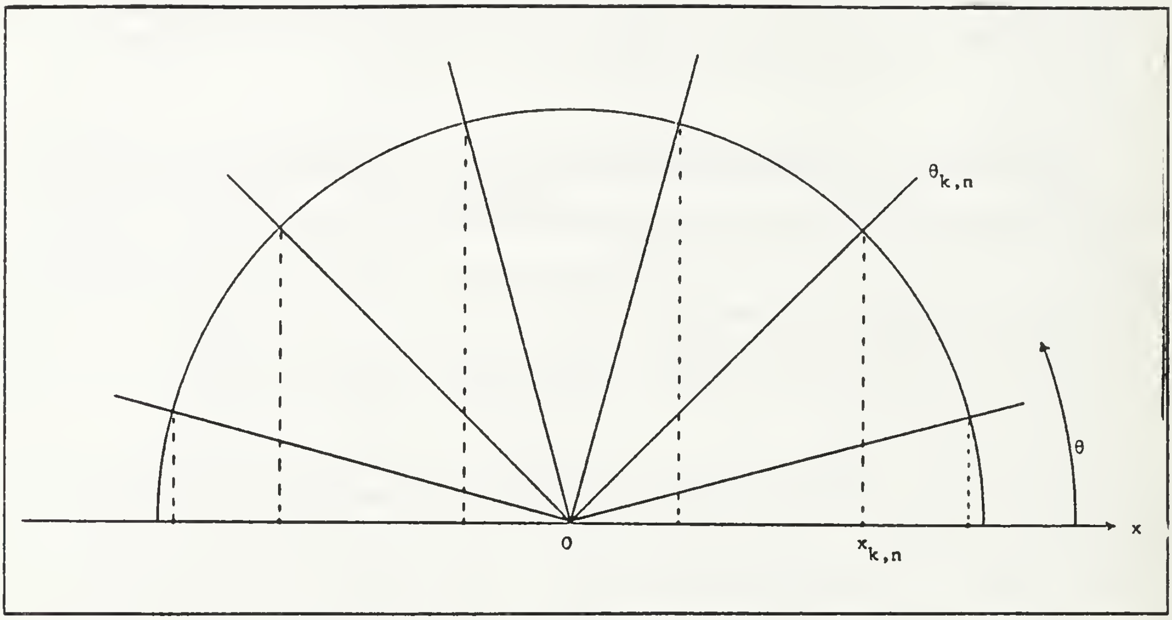

Figure 2. Zeros of Chebyshev Polynomials

In order to understand the general theory, it is first necessary to define the abstract concept of orthogonality in an appropriately defined "space" of functions. We turn our attention to these fundamental ideas in the next chapter.

II. BACKGROUND

A. ELEMENTARY LINEAR ALGEBRA

1. Vector Spaces

Let $\mathbb{R}^n$ denote the collection of all vectors (n-tuples), $\mathbf{u} = (a_1, a_2, \ldots, a_n)$, where each $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. The *standard Euclidean inner product* (also referred to as the *dot product*) of two such vectors $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{v} = (b_1, b_2, \ldots, b_n)$ is given by

$$< \mathbf{u}, \mathbf{v} > = \sum_{i=1}^{n} a_i b_i.$$

The length, or *norm,* of a vector $\mathbf{u} \in \mathbb{R}^n$ is given by

$$\|\mathbf{u}\| = \sqrt{< \mathbf{u}, \mathbf{u} >} = \left(\sum_{i=1}^{n} | a_i |^2 \right)^{\frac{1}{2}}.$$

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are perpendicular, or *orthogonal,* if and only if $< \mathbf{u}, \mathbf{v} > = 0$.

The objective of this chapter is to extend these familiar notions to objects other than classical Euclidean vectors, in particular, the "vector space" of polynomials defined on a real interval $[a, b]$.

A *vector space* V over a scalar field F (usually $\mathbb{R}$ or $\mathbb{C}$) is a nonempty set of objects called *vectors,* for which the operations of addition and scalar multiplication are defined. Addition is a rule for associating with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V an element $\mathbf{u} + \mathbf{v}$, called the *sum* of $\mathbf{u}$ and $\mathbf{v}$. Scalar multiplication is a rule for associating with each scalar c in F and each vector $\mathbf{u}$ in V an element $c\mathbf{u}$, called the *scalar multiple* of $\mathbf{u}$ by c. [Ref. 2: p. 150]

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $c, d \in F$, a vector space V must satisfy:

1. Additive closure. $\mathbf{u}, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V$
2. Commutativity. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. Associativity. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. Additive identity. There exists a *zero* vector, $\mathbf{0} \in V$, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. Additive inverse. For each $\mathbf{u} \in V$, there exists a vector $-\mathbf{u} \in V$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. Multiplicative closure. $\mathbf{u} \in V$ and $c \in F \Rightarrow c\mathbf{u} \in V$

7. Distributivity. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

8. Distributivity. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

9. Multiplicative associativity. $c(d\mathbf{u}) = (cd)\mathbf{u}$

10. Multiplicative identity. There exists a scalar $1 \in F$ such that $1\mathbf{u} = \mathbf{u}$.

Example 1: $\mathbb{R}^n$ (the model)
Example 2: $P_N[a,b] = \{$polynomials of degree $\leq N$ on the interval $[a,b]\}$
Example 3: $P[a,b] = \{$polynomials on the interval $[a,b]\}$
Example 4: $C[a,b] = \{$continuous functions on the interval $[a,b]\}$

Note that $P_N[a,b] \subset P[a,b] \subset C[a,b]$. These vector spaces are sometimes referred to as *function spaces.* The interval $[a,b]$ may be finite or infinite (i. e., $[a, \infty)$, $(-\infty, b]$, or $(-\infty, \infty)$) for our purposes.

A subset U of a vector space V is said to be a *vector subspace* of V if it is a vector space in its own right.

Example 5: $P_N[a,b]$ is a subspace of $P[a,b]$, which in turn is a subspace of $C[a,b]$.

Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space V, and scalars $c_1, c_2, \dots, c_n$, the vector $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \in V$ is said to be a *linear combination* of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist scalars $c_1, c_2, \dots, c_n$, not all equal to zero, such that the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$. (Equivalently, at least one of the vectors $\mathbf{v}_i$ can be expressed as a linear combination of the others.) Otherwise, the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is *linearly independent.* An infinite set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \dots\}$ is defined to be linearly independent if every finite subset of S is linearly independent; otherwise S is linearly dependent [Ref. 3: p. 8]. The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ are said to *span* V if every vector $\mathbf{v} \in V$ can be represented as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. In this case, we write $V = span\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ form a *basis* for V if they are linearly independent and span V. The *dimension* of V is the number of elements in any basis.

Example 6: The set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the *standard basis* for $\mathbb{R}^n$, where $\mathbf{e}_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ i.e., the vector with a one in the i^{th} position and zeros elsewhere, $i = 1, 2, \dots, n$.

Example 7: The set $\{1, x, x^2, \dots, x^N\}$ is the standard basis for $P_N[a,b]$. (Linear independence is ensured by the Fundamental Theorem of Algebra.) The dimension of $P_N[a,b]$ is therefore $N+1$.

Example 8: The set $\{1, x, x^2, \ldots, x^n, \ldots\}$ is the standard basis for $P[a,b]$, and hence $P[a,b]$ is an *infinite-dimensional* vector space.

2. Inner Product Spaces

An *inner product* on a real vector space V is a mapping

$$< \, , \, > : V \times V \to \mathbb{R}$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha, \beta \in \mathbb{R}$, the following properties hold:

1. Positive definiteness: $<\mathbf{u}, \mathbf{u}> \; \geq 0$, and $<\mathbf{u}, \mathbf{u}> \; = 0$ if and only if $\mathbf{u} = 0$
2. Symmetry: $<\mathbf{u}, \mathbf{v}> \; = \; <\mathbf{v}, \mathbf{u}>$
3. Bilinearity: $<\alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{w}> \; = \alpha <\mathbf{u}, \mathbf{w}> + \beta <\mathbf{v}, \mathbf{w}>$

A vector space with an inner product is known as an *inner product space.*

Example 9: $V = \mathbb{R}^n$; let constant "weights" $w_i > 0$ be given, $i = 1, 2, \ldots, n$. For $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{v} = (b_1, b_2, \ldots, b_n)$, $\mathbf{u}, \mathbf{v} \in V$,

$$<\mathbf{u}, \mathbf{v}> \; = \sum_{i=1}^{n} a_i \, b_i \, w_i$$

If $w_i = 1$ for $i = 1, 2, \ldots, n$, then this reduces to the standard Euclidean inner product, or dot product. Otherwise, this is referred to as a *weighted* inner product.

The next two examples are commonly applied inner products on function space, and are analogues of the previous example. We assume a given *weight function* $w(x) > 0$ in (a,b), integrable in the first case (e.g., continuous for $[a,b]$ a finite interval).

Example 10: $V = P_N[a,b]$, $P[a,b]$, $C[a,b]$

$$<f, g> \; = \int_a^b f(x)g(x)w(x)dx$$

Example 11: $V = P_N[a,b]$

$$<f, g> \; = \sum_{x=0}^{N} f(x)g(x)w(x)$$

(Positive definiteness is ensured by the Fundamental Theorem of Algebra.)

The *norm induced by the inner product* is given by $\|\mathbf{u}\| = \sqrt{<\mathbf{u}, \mathbf{u}>}$ .2

Example 12: For the inner products of Examples 10 and 11 therefore

$$\|f\| = \left(\int_a^b [f(x)]^2 \, w(x) \, dx \right)^{1/2} \quad \text{and}$$

$$\|f\| = \left(\sum_{x=0}^{N} [f(x)]^2 \, w(x) \right)^{1/2},$$

respectively. These are sometimes referred to as "L^2-norms."

Two vectors $\mathbf{u}, \mathbf{v} \in V$ are said to be *orthogonal,* denoted $\mathbf{u} \perp \mathbf{v}$, if and only if $<\mathbf{u}, \mathbf{v}> = 0$. The vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be *orthonormal* if $\mathbf{u} \perp \mathbf{v}$ and $\|\mathbf{u}\| = \|\mathbf{v}\| = 1$. Note that the orthogonality of vectors in a space is determined by the inner product being used.

The two examples which follow refer back to Chapter I, Section A.1.

Example 13: Formula (6) shows that the functions $\{ 1, \ \cos x, \ \cos 2x, \ldots \}$ are orthogonal on $[0, \pi]$ with respect to the uniform weight function $w(x) = 1$. A similar computation shows that the same property holds on $[-\pi, \pi]$ with respect to the weight function $w(x) = \frac{1}{\pi}$, i.e.,

$$< f, \ g > = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, g(x) \, dx.$$

One advantage of preferring this inner product over the standard one lies in the computation of norms. Using $\|f\| = \sqrt{< f, f >}$, we have $\| 1 \| = 2$ and $\| \cos nx \| = 1$ if $n \geq 1$. Hence the functions $\{ 1/2, \ \cos x, \ \cos 2x, \ldots \}$ are orthonormal on $[-\pi, \pi]$ with respect to the inner product above. Similar statements hold for the integral of a product of two sine functions on $[-\pi, \pi]$, as well as for the product of a sine and a cosine.

Example 14: By (7) the Chebyshev polynomials $\{T_n(x)\}$ form an orthogonal class with respect to the inner product of Example 10 above on $[-1,1]$ with the weight function $w(x) = (1 - x^2)^{-1/2}$.

---

2 Recall that $<\mathbf{u}, \mathbf{u}> \geq 0$. We remark that in the same way we defined *inner product* earlier, it is possible to define a general *norm* on a vector space which is not induced by an inner product.

We remark here that Examples 10 and 11 can be unified into a single inner product on a "polynomial" space V via

$$<f, g> = \int_E f(x)\, g(x)\, d\alpha(x),$$

where $d\alpha(x)$ is a positive Lebesgue-Stieltjes measure on a measurable set E possessing *finite moments*, i.e., $x^n\, d\alpha(x)$ integrable, $n = 0, 1, 2, \ldots$. In Example 10, $E = [a,b] \subset \mathbb{R}$ and $d\alpha(x) = w(x)dx$; the resulting expression is known as a *continuous* inner product, while in Example 11 the set E consists of a finite number of points $\{0, 1, \ldots, N\} \subset \mathbb{R}$, and the associated measure gives rise to a *discrete* inner product.

B. FOURIER SERIES

Let $\mathbf{v} \in V$, and U be an n-dimensional subspace of V having some *orthonormal* basis $\{ \mathbf{u}_1, \ldots, \mathbf{u}_n \}$. (Any basis can be orthonormalized via the *Gram-Schmidt* process - see next section.) The vector $\mathbf{v}$ can be resolved into a sum of two vector components:

(1) $$\mathbf{v} = (\mathbf{v} - \mathbf{w}) + \mathbf{w}$$

where $\mathbf{w} \in U$ and $(\mathbf{v} - \mathbf{w}) \perp U$. (See Figure 3.) The vector $\mathbf{w}$ is referred to as the *orthogonal projection* of $\mathbf{v}$ onto U. Since the vector $(\mathbf{v} - \mathbf{w})$ is orthogonal to every vector in U by construction, it follows that for each $j = 1, 2, \ldots, n$, $< \mathbf{v}-\mathbf{w}\,,\ \mathbf{u}_j > = 0$, or

(2) $$< \mathbf{v}\,,\ \mathbf{u}_j > = < \mathbf{w}\,,\ \mathbf{u}_j>.$$

Moreover, since it lies in U, vector $\mathbf{w}$ can be expressed as some linear combination of $\{ \mathbf{u}_1, \ldots, \mathbf{u}_n \}$:

$$\mathbf{w} = \sum_{i=1}^{n} c_i\, \mathbf{u}_i.$$

Take the inner product of both sides with $\mathbf{u}_j$ for each $j = 1, 2, \ldots, n$. From the assumption that $< \mathbf{u}_i\,,\ \mathbf{u}_j > = 0$ unless $i = j$, we have the property that

$$< \mathbf{w}\,,\ \mathbf{u}_j > = c_j < \mathbf{u}_j\,,\ \mathbf{u}_j > .$$

Thus,

(3) $$c_j = < \mathbf{v}\,,\ \mathbf{u}_j >$$

via (2) and the assumption that $< \mathbf{u}_j, \mathbf{u}_j > = \| \mathbf{u}_j \|^2 = 1$.

Thus,

$$\text{(4)} \qquad \mathbf{w} = \sum_{i=1}^{n} < \mathbf{v}, \mathbf{u}_i > \mathbf{u}_i$$

and this vector represents the "best approximation" in U to $\mathbf{v} \in V$ in the sense that of all vectors $\mathbf{z} \in U$, it is the projection vector $\mathbf{w} \in U$ which uniquely minimizes the distance $\| \mathbf{v} - \mathbf{z} \|$.

Suppose now that U is an infinite-dimensional subspace of V (also infinite-dimensional), having orthonormal basis $\{ \mathbf{u}_1, \dots, \mathbf{u}_n, \dots \}$. Then from (1) and (4)

$$\mathbf{v} = (\mathbf{v} - \mathbf{w}) + \sum_{i=1}^{n} < \mathbf{v}, \mathbf{u}_i > \mathbf{u}_i$$

we may write

$$\text{(5)} \qquad \mathbf{v} = \sum_{i=1}^{\infty} < \mathbf{v}, \mathbf{u}_i > \mathbf{u}_i$$

in the sense that

$$\text{(6)} \qquad \lim_{n \to \infty} \| \mathbf{v} - \mathbf{w} \| = 0,$$

i.e., the norm of the "residual vector" (and hence the vector itself) $\mathbf{v} - \mathbf{w} \to 0$ as $n \to \infty$. Formula (5) is known as the *generalized Fourier series* for $\mathbf{v} \in V$ with respect to the orthonormal basis $\{ \mathbf{u}_i \}_{i=1}^{\infty}$. The coefficients given in (3) are called the *generalized Fourier coefficients* of $\mathbf{v} \in V$. Statement (6) is known as the *norm convergence* property of Fourier series, and the "minimization property" mentioned above extends to this infinite-dimensional case.

Example 15: Let $V = C[a,b]$, and $\{ \phi_i(x) \}_{i=0}^{\infty}$ be an orthonormal basis of *eigenfunctions* (sometimes referred to simply as an *eigenbasis*) of V. Then $f \in V$ has a Fourier series representation

$$f(x) \sim \sum_{i=0}^{\infty} c_i \, \phi_i(x)$$

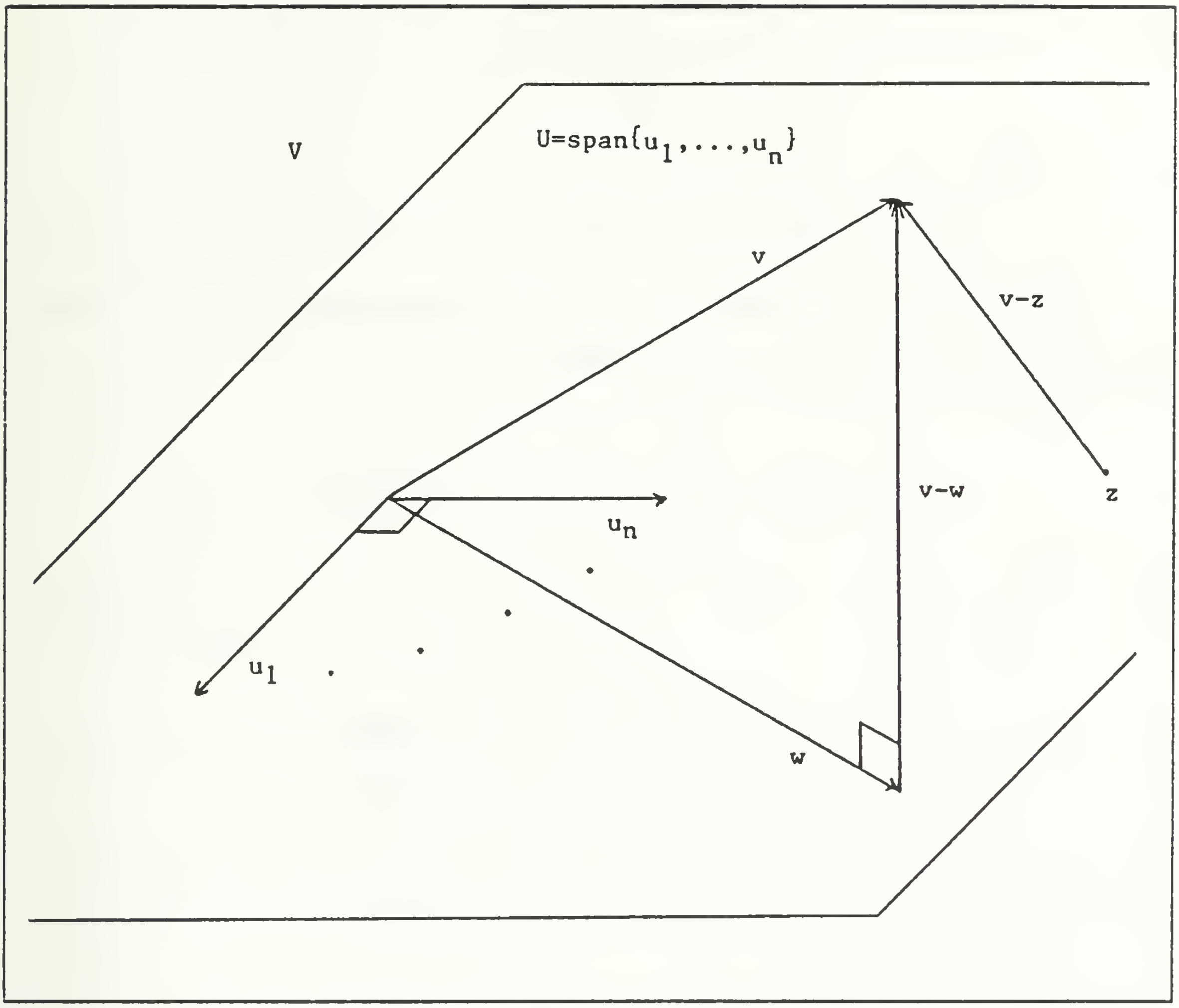

Figure 3. Best Approximation

with Fourier coefficients

$$(7) \qquad c_i = \langle f\,,\ \phi_i \rangle = \int_a^b f(x)\,\phi_i(x)\,w(x)\,dx.$$

In this function space context, norm convergence

$$\lim_{n\to\infty}\left(\int_a^b \left| f(x) - \sum_{i=0}^{n} c_i\,\phi_i(x)\right|^2 dx\right)^{1/2} = 0$$

is referred to as *mean square convergence,* and is the *least squares* principle in regression analysis.

In particular, if V is equipped with the inner product of Example 13 and orthonormal basis

$$\{ \phi_i(x) \} = \{ 1/2, \ \cos x, \ \cos 2x, \dots, \ \sin x, \ \sin 2x, \dots \}$$

on $[-\pi, \pi]$ (see Example 13, Section A.2), then a suitable function $f \in V$ (and its 2π-periodic extension on $\mathbb{R}$) has a *classical Fourier series*

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_n = < f, \ \cos nx > \ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \cos nx \, dx$$

$$b_n = < f, \ \sin nx > \ = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \ \sin nx \, dx,$$

as indicated in the Introduction.

C. GRAM-SCHMIDT ORTHONORMALIZATION

The Gram-Schmidt process orthonormalizes any set of linearly independent vectors in an inner product space. This method will be used in later sections for different inner products on the vector space $P[a,b]$.

Begin with an inner product space V and any set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \dots\}$, finite or infinite, such that any finite number of elements of this set are linearly independent. Recursively define a new set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n, \dots\}$

$$\mathbf{u}_k = \frac{\mathbf{y}_k}{\|\mathbf{y}_k\|}, \quad k = 1, 2, \dots, n, \dots$$

where $\mathbf{y}_k = \mathbf{v}_k - \mathbf{w}_k$, with[3]

---

[3] By convention, $\sum_{i=1}^{0} a_i = 0$, giving $\mathbf{w}_1 = 0$.

$$w_k = \sum_{i=1}^{k-1} < v_k \, , u_i > u_i.$$

These new vectors $\{u_1, u_2, \dots, u_n, \dots\}$ are orthonormal by construction and span the same space as the original vectors. Note that this process occurs in two stages: orthogonalization and normalization. The orthogonalization is accomplished by subtracting w_k, the orthogonal projection of v_k onto the subspace spanned by $\{u_1, u_2, \dots, u_{k-1}\}$. The component of v_k which remains, denoted above as y_k, is then orthogonal to the vectors $\{u_1, u_2, \dots, u_{k-1}\}$ as shown in Figure 4. The normalization is then achieved by dividing y_k by its norm, thus giving it unit "length".

1. Legendre Polynomials

Example 16: Let $V = P[-1,1]$ with basis $\{1, x, x^2, \dots, x^n, \dots\}$ and uniform weight function $w(x) = 1$. The inner product is then given by $< f, g > = \int_{-1}^{1} f(x)g(x)dx$. The Gram-Schmidt process yields the set

$$\{u_i\}_{i=1}^{\infty} = \{\, p_n(x)\}_{n=0}^{\infty} = \left\{ \frac{1}{\sqrt{2}} \, , \sqrt{\frac{3}{2}}\, x \, , \ \frac{3\sqrt{10}}{4} \left(x^2 - \frac{1}{3} \right) , \ \frac{5\sqrt{14}}{4} \left(x^3 - \frac{3}{5}\, x \right) , \dots \right\}$$

as an orthonormal basis for $P[-1,1]$. Since this set is linearly independent, we can standardize the set by taking scalar multiples of these polynomials so that $P_n(1) = 1$. Members of the resulting orthogonal set

$$\{P_n(x)\}_{n=0}^{\infty} = \left\{ 1, x, \frac{1}{2} \left(3x^2 - 1\right), \frac{1}{2} \left(5x^3 - 3x\right), \dots \right\}$$

are known as the *Legendre polynomials* on $[-1,1]$. If the normalized Legendre polynomials $\{\, p_n(x) \,\}_{n=0}^{\infty}$ are used as the orthogonal eigenbasis for a Fourier series, the resulting expansion is often referred to as a *Legendre series* representation; when Chebyshev polynomials are used, we obtain a *Chebyshev series* representation, etc.

The Gram-Schmidt process can always be used in this way to generate a class of orthogonal polynomials with respect to a given inner product (i.e., weight function) on a real interval. When using the Gram-Schmidt process from the basis $\{1, x, x^2, \dots, x^n, \dots\}$, the orthogonalization stage producing y_k results in a set of *monic* polynomials, i.e., the leading coefficient of each polynomial is one. In the normalization stage, we are dividing by the norm $\| y_k \| > 0$. Thus the leading coefficient of polynomials in an orthogonal class is strictly positive. In the next chapter, we will examine other

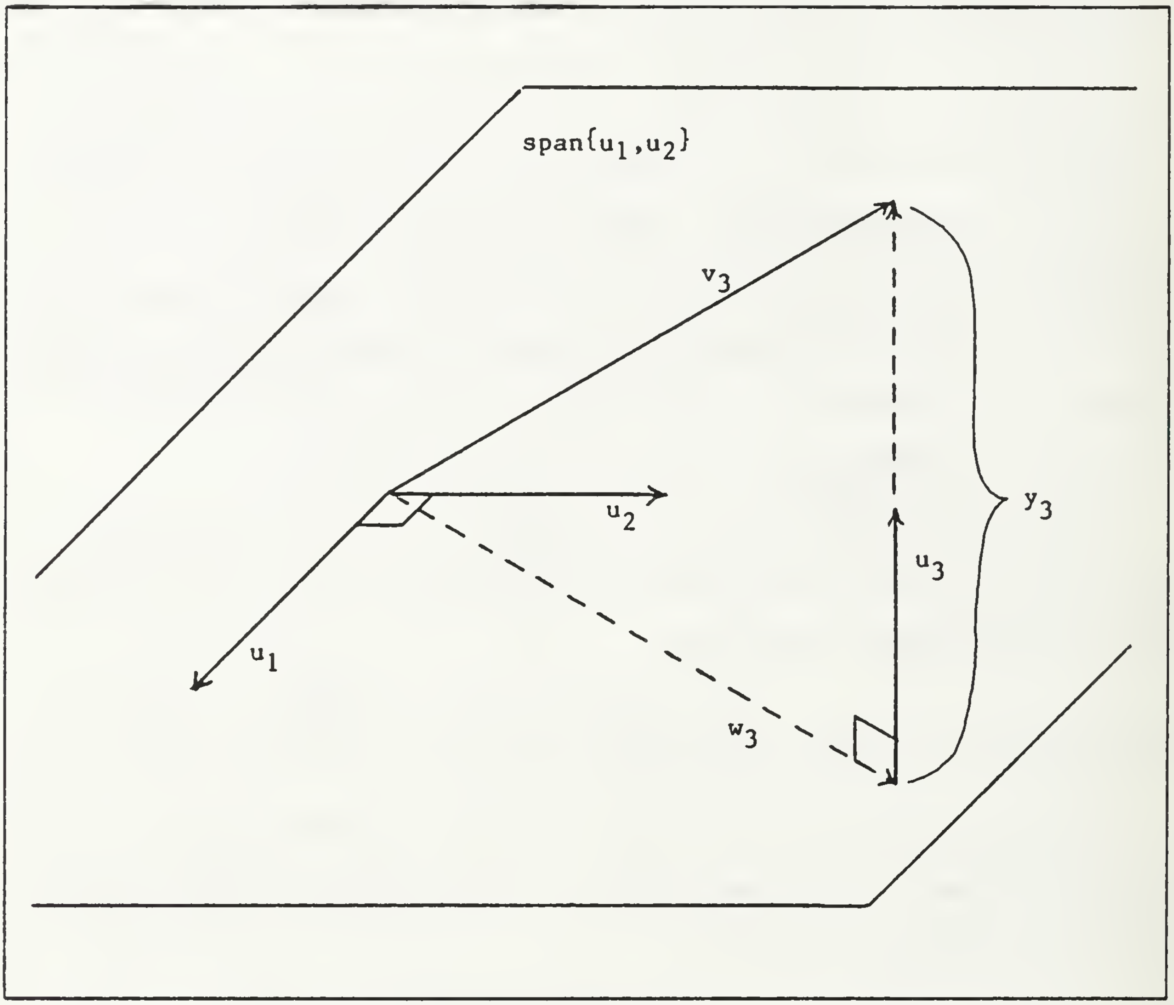

Figure 4. Orthonormalization

ways to define these classes. It is the structure and applications of certain of these classes with which we will primarily be concerned.

D. THE GAMMA FUNCTION

The *gamma function* $\Gamma(x)$ is a fundamental mathematical object that appears frequently in the representations of orthogonal polynomials as well as in many other applications. This "special function" was developed as a generalization of the factorial function of the natural numbers. As we will see, the gamma function has the value $(n-1)!$ for the positive integers n but it is defined for noninteger values as well.

A conventional definition for the gamma function is

(8) $$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \, , \; x > 0.$$

The positivity of x ensures that this improper integral converges. We now develop some fundamental properties of the gamma function. Integration by parts in (8) yields

(9) $$\Gamma(x+1) = x\Gamma(x).$$

We now introduce the *Pochhammer symbol* or *shifted factorial*, $(a)_n$, to simplify our notation. For $n > 0$, define

$$(a)_n = a(a+1)(a+2) \ldots (a+n-1) \; , \quad \text{if } n > 1$$

and $(a)_0 = 1$. Letting $a = 1$ gives $(1)_n = (1)(2)(3) \ldots (n) = n!$. Note that for a negative integer, $(-m)_n = 0$ if $n > m > 0$. The shifted factorial can be defined for negative subscripts but we will not need this in our work with polynomials.

Iteration of (9) n times yields

(10) $$\Gamma(x+n) = (x)_n \Gamma(x)$$

for every positive integer n. Using this property, the gamma function can be extended to include negative real numbers by defining

(11) $$\Gamma(x) = \frac{1}{(x)_n} \Gamma(x+n) \;\; \text{for} \;\; -n < x < -n+1.$$

Since this expression is undefined when x is zero or a negative integer, the gamma function is not defined for those values.

Letting $x = 1$ in (8) and computing directly, we have $\Gamma(1) = 1$. It then follows that $\Gamma(n+1) = n!$ by letting $x = 1$ in (10). Furthermore,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-(1/2)} dt = 2\int_0^\infty e^{-u^2} du = \sqrt{\pi}$$

where the second integral can be evaluated by standard methods involving multiple integrals.

Finally, we define a *generalized binomial coefficient* as follows. For x and α non-negative integers, define

$$\binom{x+\alpha}{x} = \frac{(x+\alpha)!}{x!\,\alpha!}\,.$$

For nonintegral α, define

$$\binom{x+\alpha}{x} = \frac{(\alpha+1)_x}{(1)_x} = \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)\,\Gamma(\alpha+1)}\,.$$

1. The Beta Function

An integral related to the gamma function defines another useful function called the *beta function* which is given by

(12)
$$B(x,\ y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

for $x\,, y > 0$. We now establish an important connection between the beta and gamma functions. We start with an identity easily verified from (12):

(13)
$$B(x,\ y+1) = B(x,\ y) - B(x+1,\ y).$$

Also from (12)

$$B(x+1,\ y) = \int_0^1 t^{x}(1-t)^{y-1}dt$$

which when integrated by parts gives

$$B(x+1,\ y) = \frac{x}{y}\,B(x,\ y+1).$$

Substituting into (13), we obtain

$$B(x,\ y) = \frac{x+y}{y}\,B(x,\ y+1)$$

which when iterated yields

$$B(x,\ y) = \frac{(x+y)_n}{(y)_n} B(x,\ y+n) = \frac{(x+y)_n}{(y)_n} \int_0^1 t^{x-1}(1-t)^{n+y-1} dt.$$

Changing variables from t to $\frac{t}{n}$,

$$B(x,\ y) = \frac{(x+y)_n}{(y)_n\, n^x} \int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^{n+y-1} dt. \tag{14}$$

Taking the limit as $n \to \infty$, and using the fact that $\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n = e^{-t}$, we have

$$B(x,\ y) = \Gamma(x) \lim_{n\to\infty} \frac{(x+y)_n}{(y)_n\, n^x}. \tag{15}$$

(The fact that we can pass the limit through the integral on the right can be mathematically justified.) If $y = 1$, then (15) gives

$$B(x,1) = \Gamma(x) \lim_{n\to\infty} \frac{(x+1)_n}{n!\, n^x}.$$

By direct evaluation using (12),

$$B(x,1) = \int_0^1 t^{x-1} dt = \frac{1}{x}.$$

Hence

$$\frac{1}{x} = \Gamma(x) \lim_{n\to\infty} \frac{(x+1)_n}{n!\, n^x},$$

which can be written as

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)_n}.$$

Noting that $x(x+1)_n = (x)_{n+1} = (x)_n(x+n)$, we have

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^{x-1} n}{(x)_n (x+n)}$$

which gives the form

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^{x-1}}{(x)_n}. \tag{16}$$

(Equation (16) was Euler's original definition of the gamma function. A separate "estimation" argument may be used to show that this limit mathematically exists.) Thus by (15),

$$B(x,\ y) = \Gamma(x) \lim_{n\to\infty} \frac{\dfrac{(x+y)_n}{n!\, n^{x+y-1}}}{\dfrac{(y)_n}{n!\, n^{y-1}}}.$$

Then by (16), we have the useful identity

$$B(x,\ y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{17}$$

We will find this identity useful in understanding the Jacobi polynomials in Chapter IV, where it becomes necessary to evaluate a related integral:

$$\int_{-1}^{1} (1-x)^{\alpha}\, (1+x)^{\beta}\, dx.$$

We remark here for future reference that the formal change of variable $x = 1-2t$ can be used to transform this integral into

$$\begin{aligned} 2^{\alpha+\beta+1} \int_0^1 t^{\alpha}\, (1-t)^{\beta}\, dt &= 2^{\alpha+\beta+1}\, B(\alpha+1,\ \beta+1) \\ &= 2^{\alpha+\beta+1}\, \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \end{aligned}$$

III. GENERAL THEORY OF CLASSICAL ORTHOGONAL POLYNOMIALS

In this chapter, we examine some of the characteristic properties associated with classes of orthogonal polynomials. Some of these properties provide alternate means of defining a class. These alternate definitions often provide a straightforward way of producing a specific result that may be very difficult to derive otherwise.

Throughout this chapter, we let $\{p_n(x)\}_{n=0}^{\infty}$ denote a set of real polynomials with $p_n(x)$ of degree n, i.e.,

(1) $$p_n(x) = k_n x^n + s_n x^{n-1} + \cdots \quad , \; k_n > 0.$$

Recall that these polynomials are said to be *orthogonal* on an interval $[a,b]$ with respect to a continuous weight function $w(x) > 0$ on (a,b) if

(2) $$< p_m , p_n > = \int_a^b p_m(x) p_n(x) w(x) dx = h_n^{-1} \delta_{m,n}$$

where the normalization $h_n^{-1} \neq 0$ is chosen to simplify the expression of certain formulas. Note that since

$$\| p_n \|^2 = \int_a^b [p_n(x)]^2 w(x) dx = h_n^{-1},$$

it follows that $h_n > 0$.

A. POLYNOMIAL EXPANSIONS

We begin by showing that any real polynomial $q_m(x)$ of degree m on $[a,b]$ can be written as a linear combination of orthogonal polynomials $\{p_n(x)\}_{n=0}^{m}$:

(3) $$q_m(x) = \sum_{i=0}^{m} \alpha_{i,m} \, p_i(x)$$

for constants $\alpha_{i,m}$, $i = 0, 1, 2, \ldots, m$.

The proof is by induction on the degree m. Since $q_m(x)$ is a real polynomial, we write

$$q_m(x) = a_m x^m + b_m x^{m-1} + \ldots$$

where $a_m \neq 0$. For $m = 0$, (3) reduces to $a_0 = \alpha_{0,0}\, k_0$ using the form for $p_0(x)$ given in (1) and $\alpha_{0,0}$ is uniquely determined.

For the induction hypothesis, suppose that for $m > 1$, we can write any polynomial of degree $m - 1$ as a linear combination of $\{ p_n(x) \}_{n=0}^{m-1}$:

$$q_{m-1}(x) = \sum_{i=0}^{m-1} \alpha_{i,m-1}\, p_i(x).$$

Since $q_m(x) - (a_m/k_m)p_m(x) = q_{m-1}(x)$ is a polynomial of degree $m - 1$ the induction hypothesis implies there is a representation

$$q_m(x) - \left(\frac{a_m}{k_m} \right) p_m(x) = \sum_{i=0}^{m-1} \alpha_{i,m-1}\, p_i(x).$$

Now set $\alpha_{m,m} = (a_m/k_m)$ and the result (3) follows. [Ref. 4: p. 33]

Using the theory of Fourier series developed earlier, we next determine the coefficients $\alpha_{i,m}$ explicitly. For $i = 0, 1, \ldots, m$, let $c_{i,m} = \alpha_{i,m}/\sqrt{h_i}$ and let

$$\phi_i(x) = \frac{p_i(x)}{\| p_i(x) \|} = \sqrt{h_i}\; p_i(x).$$

Then by construction, $\{ \phi_i(x) \}_{i=0}^{\infty}$ is an orthonormal set of polynomials. Writing (3) as

$$q_m(x) = \sum_{i=0}^{m} c_{i,m}\, \phi_i(x),$$

we see that the right-hand side can be interpreted as a (terminating) Fourier expansion of $q_m(x)$. Hence the results of Example 15 in Chapter II, Section B may be applied. In particular, by (7) in that section, the coefficients $c_{i,m}$ are given by

$$c_{i,m} = \langle q_m , \phi_i \rangle = \int_a^b q_m(x)\, \phi_i(x)\, w(x)\, dx.$$

Changing back to the old variables,

$$\alpha_{i,m} = h_i < q_m\, ,\, p_i > \; = h_i \int_a^b q_m(x)\, p_i(x)\, w(x)\, dx.$$

We are now in a position to show that the orthogonality property

$$\text{(4)} \qquad \int_a^b p_n(x) p_m(x) w(x) dx = 0 \quad , m \neq n$$

can be expressed equivalently as,

$$\text{(5)} \qquad \int_a^b p_n(x)\, x^m w(x) dx = 0\, , \;\; m < n.$$

To see (5), substitute the form (1) for $p_m(x)$ into (4) where $m < n$. The linearity of the integral gives (5). On the other hand, since x^m is a polynomial, we can write x^m as a linear combination of the orthogonal polynomials, so (5) gives (4). Note that (5) implies each $p_n(x)$ is orthogonal to every polynomial of lower degree. [Ref. 4: pp. 33-34]

B. THREE-TERM RECURRENCE RELATION

The *three-term recurrence relation* is a useful result which holds for any three consecutive orthogonal polynomials:

$$\text{(6)} \qquad p_n(x) = (A_n\, x + B_n) p_{n-1}(x) - C_n\, p_{n-2}(x) \quad , n = 2, 3, 4, \ldots$$

where A_n , B_n , and C_n are constants given by

$$A_n = \frac{k_n}{k_{n-1}} > 0 \;\; , \;\; B_n = A_n\left(\frac{s_n}{k_n} - \frac{s_{n-1}}{k_{n-1}} \right) \;\; , \;\; C_n = \frac{k_n\, k_{n-2}}{(k_{n-1})^2} \frac{h_{n-2}}{h_{n-1}} > 0.$$

The recurrence relation is valid for $n = 1$ if $p_{-1} = 0$ with C_1 arbitrary. In this case, the formula for A_n also holds for $n = 1$. (In the contrapositive form, this statement is a powerful tool for showing that a polynomial set is not orthogonal.) [Ref. 5: p. 234]

To prove this, we begin by considering $p_n(x) - (k_n/k_{n-1})\, x\, p_{n-1}(x)$, a polynomial of degree no greater than $(n - 1)$. We expand it in terms of the orthogonal polynomials $\{p_i(x)\}_{i=0}^{n-1}$ via the technique of the previous section to obtain

(7) $$p_n(x) - \frac{k_n}{k_{n-1}}\, x\, p_{n-1}(x) = \sum_{i=0}^{n-1} \alpha_{i,n-1}\, p_i(x).$$

The coefficients $\alpha_{i,n-1}$ are determined by

(8) $$\alpha_{i,n-1} = -\, h_i \frac{k_n}{k_{n-1}} < xp_{n-1}(x)\, ,\, p_i(x) > \quad , i \leq n-1$$

Because

$$< x\, p_{n-1}(x)\, ,\, p_i(x) > \; = \int_a^b x\, p_{n-1}(x)\, p_i(x)\, w(x)\, dx = \; < p_{n-1}(x)\, ,\, x\, p_i(x) > ,$$

it follows that for $i \leq n-3$, $x\, p_i(x)$ is of degree no greater than $(n-2)$. Hence

$$< p_{n-1}(x)\, ,\, x\, p_i(x) > \; = \; < x\, p_{n-1}(x)\, ,\, p_i(x) > \; = 0 \quad , i \leq n-3,$$

since $p_{n-1}(x)$ is orthogonal to every polynomial of lesser degree. Thus the constants $\alpha_{1,n-1}, \alpha_{2,n-1}, \dots , \alpha_{n-3,n-1}$ are all zero, leaving $\alpha_{n-2,n-1}$ and $\alpha_{n-1,n-1}$. With this knowledge, (7) becomes

$$p_n(x) - \frac{k_n}{k_{n-1}}\, xp_{n-1}(x) = \alpha_{n-2,n-1} p_{n-2}(x) + \alpha_{n-1,n-1} p_{n-1}(x).$$

Setting $A_n = k_n/k_{n-1}$, $B_n = \alpha_{n-1,n-1}$, and $C_n = -\, \alpha_{n-2,n-1}$ then rearranging terms gives (6).

To determine C_n explicitly, we write

$$C_n = -\, \alpha_{n-2,n-1} = A_n h_{n-2} < p_{n-1}(x)\, ,\, p_{n-2}(x) >$$

from (8). Since

$$\begin{aligned} xp_{n-2}(x) &= k_{n-2} x^{n-1} + \cdots \\ &= \frac{k_{n-2}}{k_{n-1}} \left[k_{n-1} x^{n-1} + \cdots \right] \\ &= \frac{1}{A_{n-1}} \left[p_{n-1}(x) + \sum_{j=0}^{n-2} \beta_{j,n-2}\, p_j(x) \right], \end{aligned}$$

we can write

$$C_n = h_{n-2} \frac{A_n}{A_{n-1}} < p_{n-1}(x) \, , p_{n-1}(x) + \sum_{j=0}^{n-2} \beta_{j,n-2} \, p_j(x) > .$$

Using the properties of the inner product and the orthogonality property of the polynomials, we obtain

$$C_n = h_{n-2} \frac{A_n}{A_{n-1}} < p_{n-1}(x) \, , p_{n-1}(x) > = \frac{h_{n-2}}{h_{n-1}} \frac{A_n}{A_{n-1}} = \frac{k_n k_{n-2}}{(k_{n-1})^2} \frac{h_{n-2}}{h_{n-1}}$$

as given in (6). [Ref. 6: p. 8]

Into equation (6) we substitute the expanded forms of the polynomials $p_n(x)$, $p_{n-1}(x)$, $p_{n-2}(x)$ from (1) and the constants A_n , C_n from above. Equating coefficients of x^{n-1} gives B_n. Since $k_n > 0$ and $h_n > 0$, it follows that $A_n > 0$ and $C_n > 0$ and the proof is complete.

The nonnegativity of the constants A_n and C_n is important for the converse of the result in (6). Favard showed that the existence of a three-term recurrence relation in the form of (6) implies that the polynomials of the set are orthogonal with respect to some weight function over some interval using Stieltjes integration [Ref. 7].

We observe that in order to generate the polynomials of an orthogonal class with the recurrence relation (6), we need the sequences of constants A_n , B_n , and C_n together with two of three consecutive polynomials in the class. Other techniques provide what some authors call a *pure recurrence relation* requiring only two of three consecutive polynomials to define the class, because the constants as functions of n are contained explicitly in the recurrence relation. The recurrence relation derived in Chapter I, Section A.1 for the Chebyshev polynomials is an example.

C. CHRISTOFFEL-DARBOUX FORMULA

The Christoffel-Darboux formula is an important identity which can be derived from the three-term recurrence relation. The identity is

$$\sum_{j=0}^{n} h_j \, p_j(x) p_j(y) = \frac{h_n \, k_n}{k_{n+1}} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x - y} . \tag{9}$$

To prove (9), note that from (6) we have

$$p_{j+1}(x) = (A_{j+1} x + B_{j+1}) p_j(x) - C_{j+1} p_{j-1}(x)$$

which we rearrange to give

(10) $$xp_j(x) = \frac{1}{A_{j+1}} p_{j+1}(x) - \frac{B_{j+1}}{A_{j+1}} p_j(x) + \frac{C_{j+1}}{A_{j+1}} p_{j-1}(x)$$

and similarly

(11) $$yp_j(y) = \frac{1}{A_{j+1}} p_{j+1}(y) - \frac{B_{j+1}}{A_{j+1}} p_j(y) + \frac{C_{j+1}}{A_{j+1}} p_{j-1}(y).$$

These recurrence relations are valid for $j = 0$ if we set $C_1 = 0$ and $p_{-1}(x) = p_{-1}(y) = 0$.

Multiply (10) through by $p_j(y)$ and (11) by $p_j(x)$, subtract the results, and then multiply through by h_j to obtain

$$\begin{aligned} h_j\,(x-y)\,p_j(x)\,p_j(y) = {} & \frac{h_j\,k_j}{k_{j+1}}\left[\,p_{j+1}(x)p_j(y) - p_{j+1}(y)p_j(x)\right] \\ & + \frac{h_{j-1}\,k_{j-1}}{k_j}\left[\,p_{j-1}(x)p_j(y) - p_{j-1}(y)p_j(x)\right]. \end{aligned}$$

Summing over j from 0 to n yields a telescoping series

$$(x-y)\sum_{j=0}^{n} h_j\,p_j(x)p_j(y) = \frac{h_n\,k_n}{k_{n+1}}\left[\,p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)\right]$$

from which the identity (9) follows. [Ref. 4: p. 39]

Now subtract and add the quantity $p_n(x)\,p_{n+1}(x)$ to the numerator of the right-hand side of (9) and let y tend to x to obtain a limiting case of the Christoffel-Darboux formula:

(12) $$\sum_{j=0}^{n} h_j\left[\,p_j(x)\,\right]^2 = \frac{h_n\,k_n}{k_{n+1}}\left[\,p'_{n+1}(x)\,p_n(x) - p'_n(x)\,p_{n+1}(x)\,\right].$$

We will use (12) in the next section.

D. ZEROS OF ORTHOGONAL POLYNOMIALS

In Chapter I, we observed that the zeros of the Chebyshev polynomials $\{T_n(x)\}$ are real, distinct, and lie in the interval $(-1,1)$. This principle extends to any class of polynomials orthogonal on the real interval $[a,b]$.

To see this, choose $n > 0$ and suppose that $p_n(x)$ is of constant sign in $[a,b]$. Then $< p_n(x)\,,\ p_0(x) > \neq 0$, which contradicts the assumed orthogonality. Thus by continuity (and the Intermediate Value Theorem) there exists a zero $x_1 \in (a,b)$.

Suppose that x_1 is a double root. Then $p_n(x)/(x - x_1)^2$ would be a polynomial of degree $(n - 2)$ and so

$$0 = < p_n(x)\,, p_n(x)/(x - x_1)^2 > \; = \; < 1\,,\ (p_n(x)/(x - x_1))^2 > \; > 0$$

which is a contradiction. Thus the zeros are simple.

Now suppose that $p_n(x)$ has exactly j zeros $x_1, x_2, \ldots, x_j \in (a,b)$. Then

$$p_n(x)(x - x_1)(x - x_2) \ldots (x - x_j) = q_{n-j}(x)(x - x_1)^2(x - x_2)^2 \ldots (x - x_j)^2$$

where $q_{n-j}(x)$ does not change sign in (a,b) and

$$< p_n(x)\,, (x - x_1)(x - x_2) \ldots (x - x_j) > \; = \; < q_{n-j}(x)\,, (x - x_1)^2(x - x_2)^2 \ldots (x - x_j)^2 > .$$

Since

$$< q_{n-j}(x)\,, (x - x_1)^2(x - x_2)^2 \ldots (x - x_j)^2 > \; \neq 0$$

and

$$< p_n(x)\,, (x - x_1)(x - x_2) \ldots (x - x_j) > \; = 0 \quad \text{for } j < n,$$

then it must be that $j \geq n$. The Fundamental Theorem of Algebra precludes $j > n$ and so we conclude $j = n$. [Ref. 5: p. 236]

Thus all the zeros of $p_n(x)$ are real, simple, and lie in the interval (a,b), and so may be ordered

$$a < x_{1,n} < x_{2,n} < \cdots < x_{k,n} < \cdots < x_{n,n} < b.$$

The interlacing of zeros of $p_n(x)$ and $p_{n+1}(x)$ follows from the Christoffel-Darboux formula. Recalling that the leading coefficient k_n is positive for all $p_n(x)$, then (12) gives

(13) $$p'_{n+1}(x)\, p_n(x) - p'_n(x)\, p_{n+1}(x) > 0\,, \quad -\infty < x < \infty.$$

Let u and v be adjacent zeros of $p_n(x)$. Then

(14) $$p'_n(u)\, p'_n(v) < 0$$

since the zeros are simple. At these zeros, inequality (13) reduces to

$$-p'_n(u)\, p_{n+1}(u) > 0$$

and

$$-p'_n(v)\, p_{n+1}(v) > 0.$$

Multiply these two inequalities together with (14) to conclude

$$p_{n+1}(u)\, p_{n+1}(v) < 0,$$

so $p_{n+1}(x)$ has a zero between each pair of consecutive zeros of $p_n(x)$.

Now let $x_{n,n}$ denote the largest zero of $p_n(x)$. Observing that $p_n(x) \to \infty$ as $x \to \infty$, we must have $p'_n(x_{n,n}) > 0$, and so by (13)

$$p_{n+1}(x_{n,n}) < 0.$$

But $p_{n+1}(x) \to \infty$ as $x \to \infty$, so $p_{n+1}(x)$ must have a zero to the right of $x_{n,n}$. Similarly, $p_{n+1}(x)$ must have a zero to the left of $x_{1,n}$, the smallest zero of $p_n(x)$. Thus all $n+1$ zeros of $p_{n+1}(x)$ are accounted for and interlace those of $p_n(x)$ [Ref. 8]:

$$a < x_{1,n+1} < x_{1,n} < x_{2,n+1} < \cdots < x_{k,n+1} < x_{k,n} < x_{k+1,n+1} < \cdots < x_{n,n+1} < x_{n,n} < x_{n+1,n+1} < b.$$

E. GENERATING FUNCTIONS

The function $F(x,t)$ having a formal power series expansion in t

$$F(x,t) = \sum_{n=0}^{\infty} f_n(x) t^n$$

is said to be a *generating function* for the set $\{f_n(x)\}$ [Ref. 9: p. 129]. For an appropriately chosen generating function, the *generated set* $\{f_n(x)\}$ is a class of orthogonal polynomials. By defining a class in this way, properties of the polynomials can be derived from the generating function itself.

For example, consider

$$F(x,t) = \left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} f_n(x) t^n \tag{15}$$

which for fixed x is the Taylor series of $F(x,t)$ centered at $t = 0$. If we restrict x to the interval $[-1,1]$, then by considering the singular points of $F(x,t)$, we may conclude that the series is convergent for $|t| < 1$ [Ref. 4: p. 28]. Thus we can determine $f_n(x)$ for $n = 0, 1, 2, \ldots$ by

$$f_n(x) = \frac{1}{n!}\frac{\partial^n}{\partial t^n}\left[\left(1 - 2xt + t^2\right)^{-1/2}\right]_{t=0}. \tag{16}$$

Equation (16) yields $f_0(x) = 1$, $f_1(x) = x$, $f_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, etc. We also note from (16) that

$$\begin{aligned} f_n(1) &= \frac{1}{n!}\frac{\partial^n}{\partial t^n}\left[(1 - t)^{-1}\right]_{t=0} \\ &= \frac{1}{n!}\left[n!\,(1 - t)^{-n-1}\right]_{t=0} \\ &= 1. \end{aligned}$$

We now derive some basic properties of the $f_n(x)$ from the generating function in (15).

1. Recurrence Relation

Differentiating (15), we find

$$\frac{\partial F}{\partial x} = t\left(1 - 2xt + t^2\right)^{-3/2} = \sum_{n=1}^{\infty} f_n'(x)t^n \tag{17}$$

$$\frac{\partial F}{\partial t} = (x - t)\left(1 - 2xt + t^2\right)^{-3/2} = \sum_{n=1}^{\infty} n f_n(x)t^{n-1}. \tag{18}$$

Since $(x - t)\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = 0$, we have

$$(x - t)\sum_{n=1}^{\infty} f_n'(x)t^n - t\sum_{n=1}^{\infty} n f_n(x)t^{n-1} = 0$$

which becomes

$$\sum_{n=1}^{\infty} x f_n'(x)t^n - \sum_{n=1}^{\infty} n f_n(x)t^n = \sum_{n=1}^{\infty} f_n'(x)t^{n+1}.$$

Since $f_0(x) = 1$, we can start the sum on the right side at $n = 0$, then re-index so that it starts at $n = 1$ again. We then find

$$\sum_{n=1}^{\infty} [\, x f_n'(x) - n f_n(x) \,]\, t^n = \sum_{n=1}^{\infty} f_{n-1}'(x)\, t^n .$$

Equating coefficients of t^n then gives

$$x f_n'(x) - n f_n(x) = f_{n-1}'(x)\,, \quad n \geq 1. \tag{19}$$

Rewrite (17) as

$$\left(1 - 2xt + t^2\right)^{-3/2} = \sum_{n=1}^{\infty} f_n'(x) t^{n-1}\,, \quad t \neq 0. \tag{20}$$

Substituting the appropriate expressions from (20), (18), and (15), respectively, into the identity

$$\left(1 - t^2\right)\left(1 - 2xt + t^2\right)^{-3/2} - (2t)\,(x - t)\left(1 - 2xt + t^2\right)^{-3/2} = \left(1 - 2xt + t^2\right)^{-1/2}.$$

now gives

$$\left(1 - t^2\right)\sum_{n=1}^{\infty} f_n'(x) t^{n-1} - (2t)\sum_{n=1}^{\infty} n f_n(x) t^{n-1} = \sum_{n=0}^{\infty} f_n(x) t^n$$

which, upon rearranging, becomes

$$\sum_{n=1}^{\infty} f_{n+1}'(x) t^n - \sum_{n=1}^{\infty} f_{n-1}'(x) t^n - \sum_{n=1}^{\infty} 2n f_n(x) t^n = \sum_{n=1}^{\infty} f_n(x) t^n .$$

Equating coefficients of t^n and gathering terms yields

$$(2n + 1) f_n(x) = f_{n+1}'(x) - f_{n-1}'(x)\,, \quad n \geq 1. \tag{21}$$

Substituting (19) into (21) gives

$$x f_n'(x) = f_{n+1}'(x) - (n + 1) f_n(x)$$

which by a shift of index from $n \to n - 1$ becomes

(22) $$x f_{n-1}'(x) = f_n'(x) - n f_{n-1}(x)\,, \quad n \geq 2.$$

Substituting again from (19) gives

(23) $$\left(x^2 - 1\right) f_n'(x) = nx f_n(x) - n f_{n-1}(x).$$

Multiplying (19) through by $(x^2 - 1)$, we have

$$x\left(x^2 - 1\right) f_n'(x) - n\left(x^2 - 1\right) f_n(x) = \left(x^2 - 1\right) f_{n-1}'(x).$$

Now substitute for $(x^2 - 1) f_n'(x)$ and $(x^2 - 1) f_{n-1}'(x)$ from (23) and gather terms to get

(24) $$n f_n(x) = (2n - 1)\, x f_{n-1}(x) - (n - 1) f_{n-2}(x)\,, \quad n \geq 2.$$

Equation (24) is a *three-term recurrence relation* for $\{f_n(x)\}$. The advantage of this form is that beginning with $f_0(x)$ and $f_1(x)$, we can now generate any member of $\{f_n(x)\}$ by iterating (24) and thus avoid the differentiation in (16). Note that since $f_0(x) = 1$, $f_1(x) = x$, (24) implies that $\{f_n(x)\}$ is a set of polynomials. [Ref. 9: pp.159-160]

2. Ordinary Differential Equation

We continue the same line of reasoning to extract additional information about $\{f_n(x)\}$. Differentiating (22) yields

(25) $$x f_{n-1}''(x) = f_n''(x) - (n + 1) f_{n-1}'(x).$$

From (19) we have $f_{n-1}'(x)$ and, after differentiating, $f_{n-1}''(x)$. Substituting these expressions into (25) gives

$$x\left[x f_n''(x) - (n - 1) f_n'(x)\right] = f_n''(x) - (n + 1)[x f_n'(x) - n f_n(x)]$$

which when rearranged becomes a second order ordinary differential equation

(26) $$\left(1 - x^2\right) f_n''(x) - 2x f_n'(x) + n(n + 1) f_n(x) = 0.$$

The $\{f_n(x)\}$ are solutions of (26) for $n = 0, 1, 2, \ldots$. [Ref. 9: pp.160-161]

3. Orthogonality

Rewriting (26) as

(27) $$\frac{d}{dx}\left[\left(1 - x^2\right) f_n'(x)\right] + n(n + 1) f_n(x) = 0$$

we recognize the structure of a Sturm-Liouville eigenvalue problem. Since the points $x = \pm 1$ are singular points, we require that $f_n(x)$ and $f_n'(x)$ be finite as $x \to \pm 1$. From the associated theory of the singular Sturm-Liouville problem, we conclude that the $\{f_n(x)\}$ are orthogonal on the interval $[-1,1]$ with weight function $w(x) = 1$.

Alternately, we combine (26) with

$$\frac{d}{dx}\left[(1-x^2)f_m'(x)\right] + m(m+1)f_m(x) = 0$$

where $n \neq m$, to obtain

$$(28) \qquad f_m(x)\frac{d}{dx}\left[(1-x^2)f_n'(x)\right] - f_n(x)\frac{d}{dx}\left[(1-x^2)f_m'(x)\right] + [n(n+1) - m(m+1)]f_n(x)f_m(x) = 0.$$

Since

$$\begin{aligned}\frac{d}{dx}&\left[(1-x^2)\{f_m(x)f_n'(x) - f_m'(x)f_n(x)\}\right]\\ &= \frac{d}{dx}\left[f_m(x)\left\{(1-x^2)f_n'(x)\right\} - f_n(x)\left\{(1-x^2)f_m'(x)\right\}\right]\\ &= f_m(x)\frac{d}{dx}\left[(1-x^2)f_n'(x)\right] - f_n(x)\frac{d}{dx}\left[(1-x^2)f_m'(x)\right],\end{aligned}$$

we can write (28) as

$$\frac{d}{dx}\left[(1-x^2)\{f_m(x)f_n'(x) - f_m'(x)f_n(x)\}\right] + \left[n^2 - m^2 + n - m\right]f_n(x)f_m(x) = 0$$

or

$$(n-m)(n+m+1)f_n(x)f_m(x) = \frac{d}{dx}\left[(1-x^2)\{f_m'(x)f_n(x) - f_m(x)f_n'(x)\}\right].$$

Integrating from $x = -1$ to $x = 1$, we obtain

$$(n-m)(n+m+1)\int_{-1}^{1} f_n(x)f_m(x)dx = \left[(1-x^2)\{f_m'(x)f_n(x) - f_m(x)f_n'(x)\}\right]_{-1}^{1}.$$

Since $(1 - x^2) = 0$ at $x = \pm 1$, we have

$$(n-m)(n+m+1)\int_{-1}^{1} f_n(x) f_m(x)dx = 0.$$

Recalling that $n \neq m$, we conclude

$$\int_{-1}^{1} f_n(x) f_m(x)dx = 0$$

i.e., the polynomials $\{f_n(x)\}$ are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(x) = 1$. [Ref. 9: pp. 173-174]

These are the Legendre polynomials that were previously defined using the Gram-Schmidt process. Thus the $f_n(x)$ of (15) are in fact $P_n(x)$ and (15) may be written

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{29}$$

establishing the equivalence of the generating function definition with the Gram-Schmidt definition.

Legendre and Laplace concluded that the $P_n(x)$ in (29) were polynomials of degree n in the variable x by examining a series expansion of the function

$$(1-2xr+r^2)^{-1/2} = \sum_{n=0}^{\infty} r^n \sum_{m=0}^{[n/2]} \frac{(-1)^m\,(1/2)_{n-m}}{m!\,(n-2m)!}\,(2x)^{n-2m}.$$

They reasoned the orthogonality directly. From (29) we write

$$(1-2xr+r^2)^{-1/2} = \sum_{i=0}^{\infty} P_i(x)\, r^i$$

and

$$(1-2xs+s^2)^{-1/2} = \sum_{j=0}^{\infty} P_j(x)\, s^j.$$

Multiplying these power series together via the Cauchy product formula yields

$$\frac{1}{\sqrt{1-2xr+r^2}\,\sqrt{1-2xs+s^2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} P_k(x)\, P_{n-k}(x)\, r^k\, s^{n-k}$$
$$= \sum_{n=0}^{\infty}\left[\sum_{i+j=n} P_i(x)\, P_j(x)\right] r^i\, s^j.$$

Integrating from -1 to 1 with respect to x gives

$$\int_{-1}^{1}\frac{dx}{\sqrt{1-2xr+r^2}\,\sqrt{1-2xs+s^2}} = \sum_{n=0}^{\infty}\sum_{i+j=n}\left[\int_{-1}^{1} P_i(x)\, P_j(x)\, dx\right] r^i\, s^j.$$

Through tedious calculation, the left-hand side of this expression becomes

$$\frac{1}{\sqrt{rs}}\ \log\frac{1+\sqrt{rs}}{1-\sqrt{rs}},$$

a function of the product rs. From this we conclude that the coefficients of the terms in the series on the right-hand side are zero when i and j differ, i.e.,

$$\int_{-1}^{1} P_i(x)\, P_j(x)\, dx = 0\ ,\ \ i \neq j$$

and the orthogonality is established. Finally,

$$\left(1-2xr+r^2\right)^{-1/2}\Big|_{x=1} = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

or $P_n(1) = 1$ and so the $P_n(x)$ are in fact Legendre polynomials. [Ref. 8]

In a similar fashion, the norm of the Legendre polynomials can also be obtained from the generating function. Let

$$c_n = h_n^{-1} = \int_{-1}^{1} [\, P_n(x)\,]^2\, dx.$$

Square both sides of (29) and integrate with respect to x on $[-1,1]$:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\int_{-1}^{1} P_m(x)\, P_n(x)\, dx\right) t^{m+n} = \int_{-1}^{1} \frac{1}{1-2xt+t^2}\, dx.$$

By orthogonality, the left-hand side is zero unless $m = n$, while the right-hand side is integrable in closed form:

$$\begin{aligned}\sum_{n=0}^{\infty} c_n\, t^{2n} &= \frac{-1}{2t} \ln\left(1-2xt+t^2\right)\Big|_{-1}^{1} \\ &= \frac{1}{t} \ln\left(\frac{1+t}{1-t}\right).\end{aligned}$$

This function can be expressed as a difference of two logarithms, each of which has a convergent Maclaurin series expansion in $(-1,1)$. When combined this yields

$$\sum_{n=0}^{\infty} c_n\, t^{2n} = \sum_{n=0}^{\infty} \left(\frac{2}{2n+1}\right) t^{2n}.$$

Comparing coefficients of t^{2n} on both sides gives $c_n = \dfrac{2}{2n+1}$.

Since there is no systematic theory for determining generating functions, finding one for a polynomial class can be a problem. The work above bears this out. Unfortunately, the proofs above do not easily generalize to related classes. With this in mind, let us summarize the key steps in proving that (15) is a generating function for the Legendre polynomials. First we established that $f_n(1) = 1$ for $n \geq 0$. Next, we showed that

$$\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} f_n(x)\, t^n$$

for $f_n(x)$ a polynomial of degree n in the variable x. Finally, we showed that these polynomials were orthogonal on the interval $[-1,1]$

$$\int_{-1}^{1} f_n(x) f_m(x)\, dx = 0\ , \quad m \neq n.$$

These three points are sufficient to show that the generated class is the Legendre class of orthogonal polynomials.

We now provide an alternate proof attributed to Hermite. Our interest in this proof is mainly the technique which suggests a method of generalization that we will take advantage of in the next chapter.

We begin with (15) reproduced below

$$\left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} f_n(x)\, t^n.$$

Multiply both sides of this equation by x^k and integrate from -1 to 1 to obtain

$$\int_{-1}^{1} \frac{x^k\, dx}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} t^n \int_{-1}^{1} x^k f_n(x)\, dx. \tag{30}$$

Now change variables from x to y via

$$\left(1 - 2xt + t^2\right)^{-1/2} = 1 - ty$$

giving

$$x = \frac{t\left(1 - y^2\right)}{2} + y, \quad dx = (1 - ty)dy.$$

The left-hand side of (30) becomes

$$\int_{-1}^{1} \frac{\left[\frac{1}{2}\, t\left(1 - y^2\right) + y\right]^k}{(1 - ty)} (1 - ty)\, dy$$

or simply

$$\int_{-1}^{1} \left[\frac{1}{2}\, t\left(1 - y^2\right) + y\right]^k dy.$$

Expanding the integrand of this last expression via the Binomial Theorem, we obtain

$$\sum_{j=0}^{k} \binom{k}{j} \int_{-1}^{1} t^j\, \frac{\left(1 - y^2\right)^j}{2^j}\, y^{k-j}\, dy.$$

When written as

$$\sum_{j=0}^{k}\left(\frac{1}{2^j} \binom{k}{j} \int_{-1}^{1} (1-y^2)^j y^{k-j}\, dy \right) t^j,$$

we can identify the form as a polynomial of degree k in the variable t. Comparing this form to the right-hand side of (30), we conclude that

$$\int_{-1}^{1} x^k f_n(x)\, dx = 0\,, \quad \text{for } n > k,$$

i. e., $f_n(x)$ is orthogonal to all polynomials of lower degree. The same argument as before gives $f_n(1) = 1$ for $n \geq 0$ and the proof is complete.

F. HYPERGEOMETRIC SERIES

The term "hypergeometric" was used in 1655 to distinguish a series that was "beyond" the ordinary geometric series $1 + x + x^2 + \cdots$. In 1812, Gauss presented the power series

$$1 + \frac{ab}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots$$

$c \neq 0, -1, -2, \ldots$ which is known as Gauss' series or the *ordinary hypergeometric series* [Ref. 10].

Convergence of this series for $|x| < 1$ follows directly from the Ratio test. By Raabe's test, convergence can be shown for $|x| = 1$ when $(c - a - b) > 0$ [Ref. 11: p. 5]. Gauss also introduced the notation ${}_2F_1[a,b\,;c\,;x]$ for this series. Note that ${}_2F_1[a,b\,;c\,;x]$ may be considered as much a function of four variables as a series in x. [Ref. 12: p. 1]

With the shifted factorial, the ordinary hypergeometric series can be expressed

$${}_2F_1[a,b\,;c\,;x] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} x^n.$$

Below are some examples of important functions which can be expressed as ordinary hypergeometric series.

Example 1: $\log(1 + x) = x\,{}_2F_1[1, 1\,;\, 2\,;\; -x]$

Example 2: $\sin^{-1}(x) = x\,{}_2F_1[1/2, 1/2\,;\, 3/2\,;\, x^2]$

Example 3: $\tan^{-1}(x) = x\,{}_2F_1[1/2, 1\,;\, 3/2\,;\; -x^2]$

The *generalized hypergeometric series* is formed by extending the number of parameters, an idea attributed to Clausen [Ref. 11: p. 40].

$$ {}_rF_s[a_1, a_2, \ldots, a_r\,;\, b_1, b_2, \ldots, b_s\,;\, x] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_r)_n}{(b_1)_n(b_2)_n \ldots (b_s)_n\, n!}\, x^n. $$

Note that since $(a)_{n+1}/(a)_n = n + a$, a hypergeometric series $\sum_{n=0}^{\infty} c_n x^n$ is characterized by the fact that the ratio c_{n+1}/c_n of coefficients is a quotient of two polynomials in the index n, i.e., a *rational function of n.*

The Ratio test can be used to show convergence for all values of x when $r \leq s$ and for $|x| < 1$ when $r = s + 1$. When $r > s + 1$, the series diverges for all $x \neq 0$ and the function is defined only if the series terminates. The series terminates when one or more of the numerator parameters a_i is zero or a negative integer [Ref. 11: p. 45]. This is an important characteristic of the hypergeometric series that will be used later. A power series that terminates gives a polynomial which is defined for all x. In this case, the parameters $b_1, \ldots, b_s$ may be negative integers as long as the series terminates before a zero is introduced into a denominator term.

Examples of the generalized hypergeometric series include familiar functions such as:

Example 4: $(1 + x)^a = {}_1F_0[-a\,;\; -\;;\; -x]$

Example 5: $e^x = {}_0F_0[-\;;\; -\;;\, x]$

Example 6: $\sin x = x\,{}_0F_1[-\;;\, 3/2\,;\; -x^2/4]$

Example 7: $\cos x = {}_0F_1[-\;;\, 1/2\,;\; -x^2/4]$

Example 8: The Bessel function of order α

$$ J_\alpha(x) = \frac{(x/2)^\alpha\,{}_0F_1[-\;;\, \alpha + 1\,;\; -x^2/4]}{\Gamma(\alpha + 1)} $$

where dashes indicate the absence of parameters, i.e., when $r = 0$ or $s = 0$. We will also use a common alternate notation

$$ {}_rF_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; x\right] $$

for either the ordinary or generalized hypergeometric series. [Ref. 12: p. 4]

1. Chu-Vandermonde Sum

The Chu-Vandermonde sum

$$ {}_2F_1[-n, a; c; 1] = \frac{(c-a)_n}{(c)_n} $$

is one of many useful summation formulas. Since this one will be used in a later chapter, the proof is provided below.

Basically, this is a consequence of the General Binomial theorem

$$ (1-x)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{(1)_k} x^k. $$

Starting with the identity $(1-x)^{-a}(1-x)^{-b} = (1-x)^{-a-b}$, expand both sides. Using the Cauchy product on the left side and the General Binomial theorem on the right, we have

$$ \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(a+b)_n}{(1)_n} x^n, $$

where $c_n = \sum_{k=0}^{n} \frac{(a)_k (b)_{n-k}}{(1)_k (1)_{n-k}}$. Equating coefficients of x^n, we have

$$ \sum_{k=0}^{n} \frac{(a)_k (b)_{n-k}}{(1)_k (1)_{n-k}} = \frac{(a+b)_n}{(1)_n}. $$

In order to express the left side as an ordinary hypergeometric series, ${}_2F_1$, multiply both sides by $(1)_n$ and use the identity $(1)_n/(1)_{n-k} = (-1)^k(-n)_k$ to obtain

$$ \sum_{k=0}^{n} \frac{(-n)_k (a)_k}{(1)_k} (-1)^k (b)_{n-k} = (a+b)_n. \tag{31} $$

Next,

$$
\begin{aligned}
(-1)^k(b)_{n-k} &= (-1)^n(-1)^{n-k}(b)_{n-k} \\
&= (-1)^n(-1)^{n-k}(b)(b+1)\dots(b+n-k-1) \\
&= (-1)^n(-b)(-b-1)\dots(-b-n+k+1)\left[\frac{(-b-n+1)_k}{(-b-n+1)_k}\right] \\
&= (-1)^n\frac{(-b-n+1)_n}{(-b-n+1)_k}.
\end{aligned}
$$

Using the above result, equation (31) becomes

$$
\sum_{k=0}^{n}\frac{(-n)_k(a)_k}{(1)_k}\frac{1}{(-b-n+1)_k} = (-1)^n\frac{(a+b)_n}{(-b-n+1)_n}.
$$

Let $c = -b - n + 1$, and substitute to get

$$
\begin{aligned}
{}_2F_1[-n, a; c; 1] &= (-1)^n\frac{(-c+a-n+1)_n}{(c)_n} \\
&= \frac{(-1)^n(-c+a-n+1)\dots(-c+a)}{(c)_n} \\
&= \frac{(c-a+n-1)\dots(c-a)}{(c)_n} \\
&= \frac{(c-a)_n}{(c)_n}.
\end{aligned}
$$

IV. JACOBI POLYNOMIALS AND SPECIAL CASES

This chapter focuses on the orthogonal class known as *Jacobi polynomials*, $P_n^{(\alpha,\beta)}(x)$. Until recently, the classes of orthogonal polynomials considered "classical" were usually given to be Jacobi, Gegenbauer (also called ultraspherical), Chebyshev (of first and second kind), Legendre (also called spherical), Laguerre, and Hermite. The Jacobi polynomials hold a key position in this list since the remaining classes can be viewed as special or limiting cases of this class. Today, the classical orthogonal polynomials are taken to be special or limiting cases of either of two very general orthogonal classes known as the *Askey-Wilson polynomials* and the *q-Racah polynomials*, between which we will establish a formal equivalence. Because of their complexity, these classes are described in Chapter VI after the necessary additional theory has been developed.

The results derived in the text that follows are arranged in tabular form by class at the end of this chapter.

A. JACOBI POLYNOMIALS

1. Definition / Orthogonality

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are generated by applying the orthogonalization step of the Gram-Schmidt process to the standard basis $\{1, x, x^2, \dots\}$ of $P[-1,1]$, with respect to the weight function given by a continuous beta distribution on $[-1,1]$

$$w(x\,;\alpha,\ \beta) = (1-x)^\alpha(1+x)^\beta$$

for $\alpha > -1, \beta > -1$, i.e.,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta\, dx = \left[h_n^{(\alpha,\ \beta)}\right]^{-1} \delta_{m,n}.$$

The Jacobi polynomials can also be represented by hypergeometric series

$$(1) \qquad P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right].$$

This set of polynomials is thus standardized (as was done for the Legendre class in Chapter II, Section B) :

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}.$$

We shall demonstrate that these two characterizations of the Jacobi polynomials are indeed equivalent.

To show that the polynomials defined in (1) are in fact orthogonal with respect to $w(x;\alpha,\ \beta) = (1-x)^\alpha(1+x)^\beta$ on $[-1,1]$, it suffices to show that $P_n^{(\alpha,\beta)}(x)$ is orthogonal to one polynomial of degree m for $m = 0, 1, 2, \ldots, n-1$. This is because any $P_m^{(\alpha,\beta)}(x)$, $0 \le m \le n-1$, can be expressed as a linear combination of such polynomials. While any polynomial of degree m could be used, we choose $(1+x)^m$ for reasons that will become apparent.

To establish orthogonality, we consider

$$< P_n^{(\alpha,\beta)}(x)\,,(1+x)^m > = \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\,(1+x)^m\,(1-x)^\alpha\,(1+x)^\beta\,dx.$$

By (1), the right-hand side becomes

$$\frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\left[\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!\,2^k}\int_{-1}^{1}(1-x)^{k+\alpha}\,(1+x)^{m+\beta}\,dx\right].$$

Using the last result of Chapter II, Section C.1, the change of variable $x = 1-2t$ yields

$$\frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\left[\left(\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!\,2^k}\right)2^{k+m+\alpha+\beta+1}\,B(k+\alpha+1\,,m+\beta+1)\right]$$

$$=\frac{(\alpha+1)_n}{n!}\sum_{k=0}^{n}\left[\left(\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{k!}\right)2^{m+\alpha+\beta+1}\,\frac{\Gamma(k+\alpha+1)\,\Gamma(m+\beta+1)}{(\alpha+1)_k\,\Gamma(k+m+\alpha+\beta+2)}\right].$$

Identities for the gamma function allow us to simplify this expression to the form

$$\frac{2^{\alpha+\beta+m+1}\,\Gamma(\alpha+n+1)\,\Gamma(m+\beta+1)}{\Gamma(m+\alpha+\beta+2)}\sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(m+\alpha+\beta+2)_k\,k!}$$

which can then be written

$$\frac{2^{\alpha+\beta+m+1}\,\Gamma(\alpha+n+1)\,\Gamma(m+\beta+1)}{\Gamma(m+\alpha+\beta+2)}\;{}_2F_1\left[\begin{matrix}-n\,,\,n+\alpha+\beta+1\\ m+\alpha+\beta+2\end{matrix};1\right].$$

The Chu-Vandermonde sum from Chapter II allows us to write this as

$$\frac{2^{\alpha+\beta+m+1}\,\Gamma(\alpha+n+1)\,\Gamma(m+\beta+1)}{\Gamma(m+\alpha+\beta+2)}\;\frac{(m+1-n)_n}{(m+\alpha+\beta+2)_n}.$$

Thus,

$$< P_n^{(\alpha,\beta)}(x)\,,(1+x)^m > \;=\begin{cases}c_n\,, & m=n\\ 0\,, & m=0,1,2,\dots,n-1\end{cases}$$

where

$$c_n=\frac{n!\;2^{n+\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}$$

which justifies the orthogonality. It is possible to extract the value of $[h_n^{(\alpha,\beta)}]^{-1}$ by modifying this argument, but we defer this computation till the next section, where it will be easier.

2. Ordinary Differential Equation / Rodrigues' Formula / Norm

We begin deriving the ordinary differential equation for the Jacobi polynomials by noting the general formal result:

(2)
$$\frac{d}{dx}\;{}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};x\right]=\frac{a_1\dots a_r}{b_1\dots b_s}\;{}_rF_s\left[\begin{matrix}a_1+1,\dots,a_r+1\\ b_1+1,\dots,b_s+1\end{matrix};x\right].$$

This can be seen from the definition of the generalized hypergeometric series by writing

$$\frac{d}{dx}\,{}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};x\right]=\frac{d}{dx}\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_r)_k}{(b_1)_k\dots(b_s)_k}\;\frac{x^k}{k!}$$

$$=\sum_{k=1}^{\infty}\frac{(a_1)_k\dots(a_r)_k}{(b_1)_k\dots(b_s)_k}\;\frac{x^{k-1}}{(k-1)!}$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_r)_{k+1}}{(b_1)_{k+1}\dots(b_s)_{k+1}}\;\frac{x^k}{k!}\,.$$

Noting that $(a)_{k+1}=a\,(a+1)_k$, we have

$$\frac{d}{dx}\,{}_rF_s\left[\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};x\right]=\frac{a_1\dots a_r}{b_1\dots b_s}\sum_{k=0}^{\infty}\frac{(a_1+1)_k\dots(a_r+1)_k}{(b_1+1)_k\dots(b_s+1)_k}\;\frac{x^k}{k!}$$

from which (2) follows. Differentiation of the hypergeometric series is justified by recalling that the power series is convergent for $|\,x\,|<1$ when $r=s+1$.

We apply this result to the Jacobi polynomials to obtain

$$\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x)=\frac{d}{dx}\left(\frac{(\alpha+1)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\,n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right]\right)$$

$$=\left(\frac{-1}{2}\right)\frac{(\alpha+1)_n}{n!}\;\frac{-n\,(n+\alpha+\beta+1)}{\alpha+1}\,{}_2F_1\left[\begin{matrix}-n+1,\,n+\alpha+\beta+2\\ \alpha+2\end{matrix};\frac{1-x}{2}\right]$$

$$=\frac{(n+\alpha+\beta+1)\,(\alpha+2)_{n-1}}{2\,(n-1)!}\;\frac{(n-1)!}{(\alpha+2)_{n-1}}\;P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

which simplifies to

(3) $$\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x)=\frac{(n+\alpha+\beta+1)}{2}\;P_{n-1}^{(\alpha+1,\,\beta+1)}(x).$$

With these results established, we now consider the orthogonality property of the class: set

(4) $$I_{m,n}^{(\alpha,\beta)}=\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)\,(1-x)^{\alpha}(1+x)^{\beta}dx=\left[h_n^{(\alpha,\,\beta)}\right]^{-1}\delta_{m,n}.$$

By (3), we can write

$$I_{m,n}^{(\alpha,\beta)} = \frac{2}{m+\alpha+\beta} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta \left[\frac{d}{dx}\; P_{m+1}^{(\alpha-1,\,\beta-1)}(x) \right] dx.$$

Integration by parts yields

$$(5) \qquad I_{m,n}^{(\alpha,\beta)} = \frac{-2}{m+\alpha+\beta} \int_{-1}^{1} \left[\frac{d}{dx} \left(P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta \right) \right] P_{m+1}^{(\alpha-1,\,\beta-1)}(x)dx$$

noting that the boundary terms vanish, so that

$$(6) \qquad I_{m,n}^{(\alpha,\beta)} = \frac{-2}{m+\alpha+\beta} \int_{-1}^{1} P_{m+1}^{(\alpha-1,\,\beta-1)}(x)\, q_{n+1}(x)\,(1-x)^{\alpha-1}\,(1+x)^{\beta-1}\,dx$$

where

$$(7) \qquad q_{n+1}(x) = \left(1-x^2\right) \frac{d}{dx}\; P_n^{(\alpha,\beta)}(x) \;-\; \alpha\,(1+x)\,P_n^{(\alpha,\beta)}(x) \;+\; \beta\,(1-x)\,P_n^{(\alpha,\beta)}(x)$$

is a polynomial of degree $(n+1)$. Thus by (6) we may express $q_{n+1}(x)$ as a linear combination of Jacobi polynomials

$$q_{n+1}(x) = \sum_{j=0}^{n+1} c_j\, P_j^{(\alpha-1,\,\beta-1)}(x).$$

We would now like to show that $c_0 = c_1 = \cdots = c_n = 0$, so that only the last term of the summation survives. The constants c_j, $j = 0, 1, 2, \ldots, n$ can be determined by substituting this expression for $q_{n+1}(x)$ into (6) and using (4).

$$\begin{aligned} I_{m,n}^{(\alpha,\beta)} &= \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta\,dx \\ &= \frac{-2}{m+\alpha+\beta} \sum_{j=0}^{n+1} c_j \int_{-1}^{1} P_{m+1}^{(\alpha-1,\,\beta-1)}(x)\,P_j^{(\alpha-1,\,\beta-1)}(x)\,(1-x)^{\alpha-1}\,(1+x)^{\beta-1}\,dx. \end{aligned}$$

For each $m \le n-1$, by (4) the left-hand side $I_{m,n} = 0$, but the right-hand side is zero unless $j = m+1 \le n$. Thus

$$0 = c_{m+1}\left[h_{m+1}^{(\alpha-1,\beta-1)}\right]^{-1}, \text{ i.e.,}$$

$$c_j = 0\,,\ j \le n$$

which means from (6)

(8) $$I_{m,n}^{(\alpha,\beta)} = -\frac{2}{m+\alpha+\beta}\, c_{n+1}\, I_{m+1,n+1}^{(\alpha-1,\beta-1)}$$

and

(9) $$q_{n+1}(x) = c_{n+1}\, P_{n+1}^{(\alpha-1,\ \beta-1)}(x).$$

To determine the remaining constant c_{n+1}, we note that by (7) and (9)

$$\left(1-x^2\right)\frac{d}{dx}\, P_n^{(\alpha,\beta)}(x) - \alpha\,(1+x)\, P_n^{(\alpha,\beta)}(x) + \beta\,(1-x)\, P_n^{(\alpha,\beta)}(x) = c_{n+1}\, P_{n+1}^{(\alpha-1,\ \beta-1)}(x).$$

Letting $x = 1$, then from the hypergeometric series definition of the Jacobi polynomials we have $P_n^{(\alpha,\ \beta)}(1) = \dfrac{(\alpha+1)_n}{n!}$ which gives

$$-2\alpha\,\frac{(\alpha+1)_n}{n!} = c_{n+1}\,\frac{(\alpha)_{n+1}}{(n+1)!}$$

and so

(10) $$c_{n+1} = -2(n+1).$$

Combining (5), (6), (9), and (10), we have

(11) $$\frac{d}{dx}\left[\, P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta\,\right] = -2\,(n+1)\left[\, P_{n+1}^{(\alpha-1,\ \beta-1)}(x)\,(1-x)^{\alpha-1}\,(1+x)^{\beta-1}\right].$$

To obtain the second order ordinary differential equation, change $n \to n-1$, $\alpha \to \alpha+1$, and $\beta \to \beta+1$ in (11) to give

$$\frac{d}{dx}\left[P_{n-1}^{(\alpha+1,\ \beta+1)}(x)\,(1-x)^{\alpha+1}\,(1+x)^{\beta+1}\,\right] = -2\,n\,P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta.$$

Then by (3),

$$\frac{d}{dx}\left[\frac{2}{n+\alpha+\beta+1}\frac{d}{dx}P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha+1}(1+x)^{\beta+1}\right]=-2\,n\,P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta},$$

i. e., $y=P_n^{(\alpha,\beta)}(x)$ satisfies

$$\frac{d}{dx}\left[(1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{dy}{dx}\right]=-n(n+\alpha+\beta+1)(1-x)^{\alpha}(1+x)^{\beta}y.$$

Using the product rule to expand the left-hand side, we have

$$(1-x)^{\alpha}(1+x)^{\beta}\left[-(\alpha+1)(1+x)y'+(\beta+1)(1-x)y'+\left(1-x^2\right)y''\right]$$
$$=-n(n+\alpha+\beta+1)(1-x)^{\alpha}(1+x)^{\beta}y$$

which, when simplified, becomes the second order ordinary differential equation for the Jacobi polynomials $y=P_n^{(\alpha,\beta)}(x)$:

(12) $$\left(1-x^2\right)y''+[(\beta-\alpha)-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0.$$

The reader is invited to compare this result with the second order ordinary differential equation for the Chebyshev polynomials $\{T_n(x)\}$ given by Equation (5) in Chapter I.

By iterating (11) k times, we obtain

$$\frac{d^k}{dx^k}\left[P_n^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}\right]=(-1)^k\,2^k\,(n+1)_k\left[P_{n+k}^{(\alpha-k,\,\beta-k)}(x)(1-x)^{\alpha-k}(1+x)^{\beta-k}\right].$$

Setting $\alpha\to\alpha+k$, and $\beta\to\beta+k$ gives

$$(1-x)^{\alpha}(1+x)^{\beta}P_{n+k}^{(\alpha,\beta)}(x)=\frac{(-1)^k}{2^k(n+1)_k}\frac{d^k}{dx^k}\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}P_n^{(\alpha+k,\,\beta+k)}(x)\right],$$

or equivalently in terms of the weight function,

$$w(x;\alpha,\beta)\,P_{n+k}^{(\alpha,\beta)}(x)=\frac{(-1)^k}{2^k(n+1)_k}\frac{d^k}{dx^k}\left[w(x;\alpha+k,\beta+k)\,P_n^{(\alpha+k,\,\beta+k)}(x)\right],$$

the *general Rodrigues' formula* for the Jacobi polynomials. Letting $n=0$ and noting $P_0^{(\alpha,\beta)}(x)=1$ gives

$$(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x)=\frac{(-1)^k}{2^k\,k!}\frac{d^k}{dx^k}\left[(1-x)^{\alpha+k}(1+x)^{\beta+k}\right],$$

or

$$w(x\,;\alpha,\ \beta)\,P_k^{(\alpha,\ \beta)}(x) = \frac{(-1)^k}{2^k\,k!}\ \frac{d^k}{dx^k}\ [\,w(x\,;\alpha+k,\ \beta+k)\,],$$

the *classical Rodrigues' formula* for the Jacobi polynomials.

Finally, we can use these ideas to obtain the value of $[h_n^{(\alpha,\beta)}]^{-1}$. By direct computation,

$$\begin{aligned}[h_0^{(\alpha,\beta)}]^{-1} &= \int_{-1}^{1} (1-x)^{\alpha}\,(1+x)^{\beta}\,dx \\ &= \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,.\end{aligned}$$

We will use this below. Combining (8), (10), and (4) when $m=n$ yields

$$[h_n^{(\alpha,\beta)}]^{-1} = \frac{4(n+1)}{n+\alpha+\beta}\ [h_{n+1}^{(\alpha-1,\,\beta-1)}]^{-1}.$$

Making the changes $n \to n-1$, $\alpha \to \alpha+1$, and $\beta \to \beta+1$, this may be rewritten as

$$[h_n^{(\alpha,\beta)}]^{-1} = \frac{n+\alpha+\beta+1}{4n}\ [h_{n-1}^{(\alpha+1,\,\beta+1)}]^{-1}.$$

Iterating this relation n times produces

$$\begin{aligned}[h_n^{(\alpha,\beta)}]^{-1} &= \frac{(n+\alpha+\beta+1)_n}{4^n\,n!}\ [h_0^{(\alpha+n,\,\beta+n)}]^{-1} \\ &= \frac{\Gamma(2n+\alpha+\beta+1)}{2^{2n}\,n!\,\Gamma(n+\alpha+\beta+1)}\ \frac{2^{2n+\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} \\ &= \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}\,.\end{aligned}$$

3. Generating Function

As was mentioned in Chapter III, finding a generating function for a class of orthogonal polynomials can be a challenging task. Fortunately, a generating function for the Jacobi polynomials can be found by mimicking a technique attributed to Hermite in his work with the Legendre polynomials. (See Chapter III, Section E.3.)

We seek a generating function

$$F(r,x)=\sum_{n=0}^{\infty} c_n\, P_n^{(\alpha,\beta)}(x)\, r^n \tag{13}$$

for constants c_n, $n = 0, 1, 2, \ldots$, with

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, x^k\,(1-x)^\alpha\,(1+x)^\beta\,dx = 0\,,\ \ k < n.$$

To begin, we multiply (13) by $x^k\,(1-x)^\alpha\,(1+x)^\beta$ and integrate from -1 to 1 to obtain

$$\int_{-1}^{1} x^k\,F(r,x)\,(1-x)^\alpha\,(1+x)^\beta\,dx = \sum_{n=0}^{k} c_n\left[\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\,x^k\,(1-x)^\alpha\,(1+x)^\beta\,dx\right] r^n. \tag{14}$$

Note that the summation of the right-hand side is from 0 to k because the orthogonality property makes each term zero for $n > k$. We observe that the unknowns to be determined in (14) are $F(r,x)$ and c_n. When one is given, the other can be found, so we first consider the left-hand side of (14)

$$\int_{-1}^{1} x^k\,F(r,x)\,(1-x)^\alpha\,(1+x)^\beta\,dx.$$

Setting $\sqrt{1-2xr+r^2} = 1-ry$ and substituting for $x = y + \left(\dfrac{1-y^2}{2}\right) r$ yields

$$\int_{-1}^{1}\left[y+\frac{(1-y^2)\,r}{2}\right]^k F\left(r,\ y+\frac{(1-y^2)\,r}{2}\right)\left[1-y-\frac{(1-y^2)\,r}{2}\right]^\alpha\left[1+y+\frac{(1-y^2)\,r}{2}\right]^\beta (1-ry)\,dy.$$

Factoring out the terms $(1-y)^\alpha$ and $(1+y)^\beta$, we have

$$\begin{aligned}\int_{-1}^{1}\left[y+\frac{(1-y^2)\,r}{2}\right]^k (1-y)^\alpha\,(1+y)^\beta\, F\left(r,y+\frac{(1-y^2)\,r}{2}\right)&\\ \times\left[1-\frac{(1+y)\,r}{2}\right]^\alpha\left[1+\frac{(1-y)\,r}{2}\right]^\beta (1-ry)\,dy.&\end{aligned} \tag{15}$$

We would now like to make a judicious choice of F in order to facilitate calculations. Following the lead of the Legendre polynomials (Chapter III, Section E.3) suppose F is such that

$$(16) \qquad F\left(r, y+\frac{(1-y^2)\,r}{2}\right)\left[1-\frac{(1+y)\,r}{2}\right]^{\alpha}\left[1+\frac{(1-y)\,r}{2}\right]^{\beta}(1-ry)=1,$$

then the integral in (15) becomes

$$\int_{-1}^{1}\left[y+\frac{(1-y^2)\,r}{2}\right]^{k}(1-y)^{\alpha}(1+y)^{\beta}\,dy.$$

We note that the integrand is a polynomial of degree k in the variable r as is the right-hand side of (14). From (16), we get

$$F\left(r, y+\frac{(1-y^2)\,r}{2}\right)=(1-ry)^{-1}\left[1-\frac{(1+y)\,r}{2}\right]^{-\alpha}\left[1+\frac{(1-y)\,r}{2}\right]^{-\beta}$$

which becomes, via $1-2xr+r^2=(1-ry)^2$, the generating function for the Jacobi polynomials:

$$F(r,x)=2^{\alpha+\beta}\left(1-2xr+r^2\right)^{-1/2}\left[1-r+\left(1-2xr+r^2\right)^{1/2}\right]^{-\alpha}\left[1+r+\left(1-2xr+r^2\right)^{1/2}\right]^{-\beta}.$$

Using this generating function in (13) gives

$$\sum_{n=0}^{\infty} c_n\, P_n^{(\alpha,\beta)}(x)\, r^n = 2^{\alpha+\beta}\left(1-2xr+r^2\right)^{-1/2}\left[1-r+\left(1-2xr+r^2\right)^{1/2}\right]^{-\alpha}\left[1+r+\left(1-2xr+r^2\right)^{1/2}\right]^{-\beta}$$

from which we can determine c_n by setting $x=1$ and recalling that $P_n^{(\alpha,\beta)}(1)=\dfrac{(\alpha+1)_n}{n!}$. Thus,

$$\sum_{n=0}^{\infty} c_n\,\frac{(\alpha+1)_n}{n!}\,r^n = 2^{\alpha+\beta}(1-r)^{-1}\left[2^{-\alpha}(1-r)^{-\alpha}\right]2^{-\beta}=(1-r)^{-\alpha-1}$$

which by the Binomial Theorem becomes

$$\sum_{n=0}^{\infty} c_n\,\frac{(\alpha+1)_n}{n!}\,r^n=\sum_{n=0}^{\infty}\frac{(\alpha+1)_n}{n!}\,r^n$$

and so $c_n=1$ for all n. [Ref. 8]

B. SPECIAL AND LIMITING CASES

With the structure of the Jacobi polynomials established, we turn now to the role of the parameters α and β.

1. Special Cases

For certain choices of the parameters α and β, we find that the classes previously examined and several new classes are produced as special cases of the Jacobi class. These subclasses inherit the structure of the parent class which often provides a direct way to establish specific properties (i.e., polynomial nature, orthogonality, etc.)

Table 1 provides the choice of parameters for selected classes.

Table 1. PARAMETERS FOR SELECTED CLASSES

Class	Parameters
Jacobi	$\alpha > -1, \quad \beta > -1$
Gegenbauer	$\alpha = \beta = \lambda - 1/2$
Chebyshev, First Kind	$\alpha = \beta = -1/2$
Chebyshev, Second Kind	$\alpha = \beta = 1/2$
Legendre	$\alpha = \beta = 0$

2. Limiting Cases

In this section, we briefly examine the Laguerre polynomials and the Hermite polynomials. Using the hypergeometric series definition of the Jacobi polynomials, we show how the Laguerre class is a limiting case of the Jacobi class.

a. Laguerre Polynomials

The *Laguerre polynomials*, $L_n^{(\alpha)}(x)$, are defined in terms of hypergeometric series

(17) $$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} {}_1F_1\left[\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right].$$

The first relationship to establish is

$$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right).$$

To show this, we begin with the hypergeometric series representation for the Jacobi polynomials

$$P_n^{(\alpha,\,\beta)}(y) = \binom{n+\alpha}{n}\,{}_2F_1\left[\begin{matrix}-n,\, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-y}{2}\right].$$

When $y = 1-\dfrac{2x}{\beta}$, we have

$$P_n^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right) = \binom{n+\alpha}{n}\,{}_2F_1\left[\begin{matrix}-n,\, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{x}{\beta}\right].$$

Writing out the power series, we obtain

$$\begin{aligned} P_n^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right) &= \binom{n+\alpha}{n}\sum_{k=0}^{\infty}\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!}\,\frac{x^k}{\beta^k}\\ &= \binom{n+\alpha}{n}\sum_{k=0}^{\infty}\frac{(-n)_k\,x^k}{(\alpha+1)_k\,k!}\,\frac{(n+\alpha+\beta+1)_k}{\beta^k}\,. \end{aligned}$$

In the limit as $\beta \to \infty$, the ratio

$$\frac{(n+\alpha+\beta+1)_k}{\beta^k} \to 1$$

and so

$$\lim_{\beta\to\infty} P_n^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right) = \binom{n+\alpha}{n}\sum_{k=0}^{\infty}\frac{(-n)_k}{(\alpha+1)_k\,k!}\,x^k = L_n^{(\alpha)}(x)$$

completing the proof. [Ref. 13: p. 103]

To obtain the orthogonality relation for the Laguerre polynomials, we start with the orthogonality relation for the Jacobi polynomials

$$\int_{-1}^{1} P_m^{(\alpha,\,\beta)}(y)\,P_n^{(\alpha,\,\beta)}(y)\,(1-y)^{\alpha}\,(1+y)^{\beta}\,dy = 0\,,\quad m \neq n.$$

Letting $y = 1 - \frac{2x}{\beta}$ gives

$$\int_0^\beta P_m^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right) P_n^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right)\left(\frac{2x}{\beta}\right)^\alpha \left[2\left(1-\frac{x}{\beta}\right)\right]^\beta \frac{2}{\beta}\, dx = 0\,, \quad m \neq n.$$

Passing the constants outside of the integral and dividing through by them leaves us with

$$\int_0^\beta P_m^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right) P_n^{(\alpha,\,\beta)}\left(1-\frac{2x}{\beta}\right) x^\alpha \left(1-\frac{x}{\beta}\right)^\beta dx = 0\,, \quad m \neq n.$$

Taking the limit as $\beta \to \infty$, we obtain the orthogonality relation for the Laguerre polynomials

(18) $$\int_0^\infty L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(x)\, x^\alpha e^{-x}\, dx = 0\,, \quad m \neq n.$$

b. *Hermite Polynomials*

The *Hermite polynomials*, $H_n(x)$, are defined

(19) $$H_n(x) = (2x)^n\, {}_2F_0\left[\begin{matrix} -n/2\,, (-n+1)/2 \\ - \end{matrix}\,; -\frac{1}{x^2}\right].$$

In a fashion similar to that for the Laguerre polynomials, the Hermite polynomials are a limiting case of the Jacobi polynomials via the Gegenbauer polynomials. Specifically,

$$H_n(x) = n!\lim_{\lambda\to\infty} \lambda^{-n/2}\, C_n^{(\lambda)}(\lambda^{-1/2}x)$$

which allows a derivation of the orthogonality relation

(20) $$\int_{-\infty}^\infty H_m(x)\, H_n(x)\, e^{-x^2}\, dx = 0\,, \quad m \neq n$$

from that of the Jacobi class. [Ref. 13: p. 107]

C. DISCRETE EXTENSIONS

We turn now to orthogonal polynomial classes which use the discrete inner product introduced in Chapter II, Section A.2

$$< f, g > = \sum_{x=0}^{N} f(x)\, g(x)\, w(x).$$

Here instead of a continuum the support of the weight function is concentrated of a finite set of discrete mass points $\{0, 1, \ldots, N\}$. The polynomial classes of particular interest are the Hahn, dual Hahn, and the Racah polynomials.

1. Hahn Polynomials

The Hahn polynomials - actually discovered by Chebyshev - were independently realized by physicists working in angular momentum theory via 3-j, or Clebsch-Gordon coefficients [Ref. 8]. We define this class by the generalized hypergeometric series

$$Q_n(x\,;\alpha,\ \beta,\ N) = {}_3F_2\left[\begin{matrix} -n,\ -x,\ n+\alpha+\beta+1 \\ \alpha+1,\ -N \end{matrix}\ ;\ 1\right] \tag{21}$$

for $\alpha > -1$, $\beta > -1$, where N is a positive integer and $n = 0, 1, \ldots, N$. From the power series

$$Q_n(x\,;\alpha,\ \beta,\ N) = \sum_{k=0}^{n} \frac{(-n)_k\,(-x)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,(-N)_k\,k!}\ (1)^k,$$

we note that the variable x does not appear where we have come to expect. Since

$$\begin{aligned}(-x)_k &= (-x)\,(-x+1)\ \ldots\ (-x+k-1)\\ &= (-1)^k\,(x)\,(x-1)\ \ldots\ (x-k+1),\end{aligned}$$

we conclude that $Q_n(x\,;\alpha,\ \beta,\ N)$ is a polynomial of degree n in the variable x. Because $0 \le n \le N$, this set of $N+1$ orthogonal polynomials is finite for fixed α and β. [Ref. 14: p. 33]

The Hahn polynomials satisfy the discrete orthogonality relation

$$\sum_{x=0}^{N} Q_m(x\,;\alpha,\ \beta,\ N)\,Q_n(x\,;\alpha,\ \beta,\ N)\,w(x\,;\alpha,\ \beta,\ N) = 0\,,\quad m \neq n \tag{22}$$

where the weight function is given by a *hypergeometric* distribution

$$w(x\,;\alpha,\ \beta,\ N)=\binom{x+\alpha}{x}\binom{N-x+\beta}{N-x} \tag{23}$$

$$=\frac{(\alpha+1)_x}{(1)_x}\ \frac{(\beta+1)_{N-x}}{(1)_{N-x}}\,.$$

We note that the weight function may also be written

$$w(x\,;\alpha,\ \beta,\ N)=\frac{(\beta+1)_N}{(1)_N}\ \frac{(\alpha+1)_x\,(-N\,)_x}{(1)_x\,(-N-\beta)_x}\,. \tag{24}$$

We introduce the *first forward difference operator* acting on x

$$\Delta f_n(x)=f_n(x+1)-f_n(x)$$

as a discrete analogue to differentiation. Since

$$\sum_{t=0}^{x}\Delta f_n(t)=f_n(x+1)-f_n(0),$$

(a discrete analogue of the Fundamental Theorem of Integral Calculus) the first forward difference operator is a discrete inverse of the summation operator. This difference operator is used in the Rodrigues' formula for the Hahn polynomials, which can also be written in terms of the weight function $w(x\,;\alpha,\ \beta,\ N)$ with shifted parameters as was done for the Jacobi polynomials.

Two limiting cases of the Hahn polynomials are of particular interest. In the first case, replace x by Nx in the interval of orthogonality. This in effect places the support of the weight function on the $(N+1)$ equally-spaced points $\{0,\ 1/N,\ 2/N,\ \ldots\ ,\ 1\}$ in $[0,1]$. As $N\rightarrow\infty$, the set of discrete mass points tends towards the full interval $[0,1]$; we expect that this structure is reflected in the discrete orthogonality relation (22) becoming an integral orthogonality with respect to a continuous weight function on that interval. Rewriting the weight function in (23) as

$$w(x\,;\alpha,\ \beta,\ N)=\frac{\Gamma(x+1+\alpha)}{\Gamma(x+1)}\ \frac{\Gamma(N-x+1+\beta)}{\Gamma(N-x+1)}$$

and using a consequence of Stirling's Formula [Ref. 15: p. 257]

$$\lim_{t\to\infty} t^{-a} \frac{\Gamma(t+a)}{\Gamma(t)} = 1,$$

we see that

$$\Gamma(\alpha+1)\,\Gamma(\beta+1) \lim_{N\to\infty} N^{-\alpha-\beta}\, w(Nx\,;\alpha,\ \beta,\ N) = x^{\alpha}(1-x)^{\beta},$$

which we recognize as the continuous beta distribution for the Jacobi polynomials normalized on [0,1].

This is indeed the case, and can be easily verified by a direct computation on the hypergeometric series definitions (21) of $Q_n(x\,;\alpha,\ \beta,\ N)$ and (1) of $P_n^{(\alpha,\beta)}(x)$ given in Section A.1, i.e.,

$$\lim_{N\to\infty} Q_n(Nx\,;\alpha,\ \beta,\ N) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\ \beta)}(1-2x).$$

Thus the Hahn polynomials may be viewed as a discrete analogue and generalization of the Jacobi polynomials. [Ref. 14: p. 36]

The second limiting case gives rise to an interesting class of polynomials which has applications in coding theory [Ref. 16].

a. Krawtchouk Polynomials

For $0 < p < 1$, let $\alpha = pt$, $\beta = (1-p)\,t$ in (21), then take the limit as t tends to infinity to obtain the *Krawtchouk polynomials,* i.e.,

$$\lim_{t\to\infty} Q_n(x\,;pt\,,(1-p)\,t\,,N) =$$

(25)
$$ {}_2F_1\left[\begin{matrix}-n\,,-x\\ -N\end{matrix}\,;\ \frac{1}{p}\right] = K_n(x\,;p,\ N)$$

[Ref. 14: p. 38]. A similar limiting argument applied to the weight function shows that the Krawtchouk polynomials are orthogonal with respect to a *binomial* distribution:

(26)
$$\sum_{x=0}^{N} K_m(x\,;p,\ N)\,K_n(x\,;p,\ N)\binom{N}{x} p^{x}(1-p)^{N-x} = 0\,,\quad m\neq n.$$

This class of polynomials possesses an inherent symmetry, the structure of which can be generalized to form other orthogonal polynomial classes. To this end, we

turn now to the characteristic of *duality*. For suitably defined functions $\mu(x)$ and $\nu(x)$, two classes of orthogonal polynomials $\{ p_n(\mu(x)) \}$, and $\{ q_n(\nu(x)) \}$ are said to be *dual* if

$$p_x(\mu(n)) = q_n(\nu(x)).$$

That is, interchanging the roles of the degree n and the discrete variable x in one class produces the other. The Krawtchouk polynomials provide an example of an orthogonal class that is *self-dual,* i.e.,

$$K_n(x\,;p,\ N) = K_x(n\,;p,\ N).$$

This is clear from the hypergeometric series definition (25) given above. It therefore may seem reasonable to suspect that there exists a class of orthogonal polynomials dual to the Hahn polynomials. Such a class does in fact exist, and it is this dual class which we next examine. [Ref. 17: p. 657]

2. Dual Hahn Polynomials

The *dual Hahn polynomials,* $R_n(\lambda(x)\,;\gamma,\delta,\ N)$, are defined

(27) $$R_n(\lambda(x)\,;\gamma,\delta,\ N) = {}_3F_2\left[\begin{matrix} -n,\ -x,\ x+\gamma+\delta+1 \\ \gamma+1,\ -N \end{matrix}\ ;\ 1\right]$$

where $\lambda(x) = x(x+\gamma+\delta+1)$, and $n = 0, 1, \dots, N$. These polynomials satisfy the orthogonality relation

(28) $$\sum_{x=0}^{N} R_m(\lambda(x)\,;\gamma,\delta,\ N)\, R_n(\lambda(x)\,;\gamma,\delta,\ N)\, w(x\,;\gamma,\delta,\ N) = 0\,,\quad m \neq n$$

where

(29) $$\begin{aligned} w(x\,;\gamma,\delta,\ N) &= \frac{(\gamma+\delta+1)_x\,((\gamma+\delta+3)/2)_x\,(\gamma+1)_x\,(-N)_x\,(-1)^x}{(1)_x\,((\gamma+\delta+1)/2)_x\,(\delta+1)_x\,(\gamma+\delta+N+2)_x} \\ &= (-1)^x\,\frac{(\gamma+\delta+1)_x\,(\gamma+1)_x\,(-N)_x}{(1)_x\,(\delta+1)_x\,(N+\gamma+\delta+2)_x}\;\frac{2x+\gamma+\delta+1}{\gamma+\delta+1}\,. \end{aligned}$$

Note that $R_n(\lambda(x)\,;\gamma,\delta,\ N)$ is a polynomial of degree n in the "variable" $\lambda(x)$. The reason can be seen directly from the hypergeometric series

$$R_n(\lambda(x); \gamma, \delta, N) = \sum_{k=0}^{n} \frac{(-n)_k (-x)_k (x+\gamma+\delta+1)_k}{(\gamma+1)_k (-N)_k \, k!}.$$

By writing the terms $(-x)_k (x+\gamma+\delta+1)_k$ in product form, we obtain

$$\prod_{j=0}^{k-1} (-x+j) \prod_{j=0}^{k-1} (x+\gamma+\delta+j+1)$$

which becomes

$$\prod_{j=0}^{k-1} (-x+j)(x+\gamma+\delta+1+j).$$

Multiplying these factors as binomials, we have

$$\prod_{j=0}^{k-1} \left[(-x)(x+\gamma+\delta+1) + j(-x) + j(x+\gamma+\delta+1) + j^2 \right]$$

which simplifies to

$$\prod_{j=0}^{k-1} \left[-x(x+\gamma+\delta+1) + j(\gamma+\delta+1) + j^2 \right].$$

Taking only that part which depends on x yields

$$\lambda(x) = x(x+\gamma+\delta+1)$$

as given above. [Ref. 18: p. 48]

As discussed in the previous section, the discrete classes interact more naturally with difference operators than with differentiation. To accomodate the quadratic form of $\lambda(x)$, we introduce the *divided difference operator*

$$\rho f_n(\lambda(x)) = \frac{\Delta f_n(\lambda(x))}{\Delta \lambda(x)} = \frac{f_n(\lambda(x+1)) - f_n(\lambda(x))}{\lambda(x+1) - \lambda(x)}.$$

The dual interplay between the dual Hahn and Hahn polynomials follows easily from their hypergeometric definitions:

$$R_x(\lambda(n);\alpha,\ \beta,\ N) = {}_3F_2\left[\begin{matrix} -x,\ -n,\ n+\alpha+\beta+1 \\ \alpha+1,\ -N \end{matrix};1\right] = Q_n(x;\alpha,\ \beta,\ N).$$

This duality is also reflected in the recurrence relations and difference equations of the two classes. By interchanging the roles of n with x, α with γ, and β with δ, the recurrence relation for one class leads to or can be extracted from the difference equation for the other class. [Ref. 14: p. 37]

3. Racah Polynomials

Both the Hahn and dual Hahn polynomials can be unified as special cases of a single larger class, the standard notation of which is similar to that of the dual Hahn class. The *Racah polynomials,* $R_n(\lambda(x);\alpha,\ \beta,\gamma,\delta)$, are defined by the hypergeometric series

$$(30) \qquad R_n(\lambda(x);\alpha,\ \beta,\gamma,\delta) = {}_4F_3\left[\begin{matrix} -n,\ n+\alpha+\beta+1,\ -x,\ x+\gamma+\delta+1 \\ \alpha+1,\ \beta+\delta+1,\ \gamma+1 \end{matrix};1\right]$$

where $\lambda(x) = x(x+\gamma+\delta+1)$, $n = 0, 1, \dots, N$, and one of $\alpha+1$, $\beta+\delta+1$, or $\gamma+1$ equals $-N$. (Physicists understand these objects via 6-j symbols.)

The orthogonality relation for the Racah polynomials is

$$(31) \qquad \sum_{x=0}^{N} R_m(\lambda(x);\alpha,\ \beta,\gamma,\delta)\, R_n(\lambda(x);\alpha,\ \beta,\gamma,\delta)\, w(x;\alpha,\ \beta,\gamma,\delta) = 0\ , \quad m \neq n$$

where

$$(32) \qquad \begin{aligned} w(x;\alpha,\ \beta,\gamma,\delta) &= \frac{(\gamma+\delta+1)_x\,((\gamma+\delta+3)/2)_x\,(\alpha+1)_x\,(\beta+\delta+1)_x\,(\gamma+1)_x}{(1)_x\,((\gamma+\delta+1)/2)_x\,(\gamma+\delta-\alpha+1)_x\,(\gamma-\beta+1)_x\,(\delta+1)_x} \\ &= \frac{(\gamma+\delta+1)_x\,(\alpha+1)_x\,(\beta+\delta+1)_x\,(\gamma+1)_x}{(1)_x\,(\gamma+\delta-\alpha+1)_x\,(\gamma-\beta+1)_x\,(\delta+1)_x}\ \frac{2x+\gamma+\delta+1}{\gamma+\delta+1} \end{aligned}$$

[Ref. 19: p. 24]. Note that $w(x;\alpha,\ \beta,\gamma,\delta)$ has a "well-poised" structure. This means that the pairwise sum of numerator and denominator parameters is constant, i.e.,

$$(\gamma+\delta+1) + (1) = \gamma+\delta+2$$
$$\left(\frac{\gamma+\delta+3}{2}\right) + \left(\frac{\gamma+\delta+1}{2}\right) = \gamma+\delta+2$$
$$(\alpha+1) + (\gamma+\delta-\alpha+1) = \gamma+\delta+2$$
$$(\beta+\delta+1) + (\gamma-\beta+1) = \gamma+\delta+2$$
$$(\gamma+1) + (\delta+1) = \gamma+\delta+2.$$

The same could be said for the weight function of the dual Hahn polynomials. (To say that a hypergeometric series is *well-poised* means that for

$${}_{r+1}F_r\left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; x\right],$$

the parameters a and b satisfy the following relation

$$a_1+1 = a_2+b_1 = \cdots = a_{r+1}+b_r.$$

Well-poisedness is an important property of certain summable hypergeometric series.)

From the hypergeometric series definition (30) of the Racah polynomials, we find that when the roles of x and n, α and γ, and β and δ are all interchanged, the series is unchanged. Thus, like the Krawtchouk polynomials, the Racah polynomials are self-dual.

To recover the Hahn polynomials as a limiting case of the Racah polynomials, let $\gamma+1 = -N$ and $\delta \to \infty$. Thus by formulas (30) and (21),

$$\lim_{\delta\to\infty} {}_4F_3\left[\begin{matrix} -n, & n+\alpha+\beta+1, & -x, & x+\gamma+\delta+1 \\ & \alpha+1, & \beta+\delta+1, & -N \end{matrix} ; 1\right] = {}_3F_2\left[\begin{matrix} -n, & n+\alpha+\beta+1, & -x \\ & \alpha+1, & -N \end{matrix} ; 1\right]$$

and then by formulas (32) and (24),

$$\lim_{\delta\to\infty} w(x\,;\alpha,\ \beta, \gamma, \delta) = \frac{(1)_N}{(\beta+1)_N}\, w(x\,;\alpha,\ \beta,\ N).$$

Likewise, by letting $\alpha+1 = -N$ and $\beta \to \infty$, we obtain the dual Hahn polynomials.

Figures 5 and 6 together with Tables 2 and 3 provide a hierarchy of classes discussed in this chapter. Tables 4-15 summarize information about selected classes in this hierarchy [Refs. 14, 12, 20, 21 , 5, 13, 9 , 22].

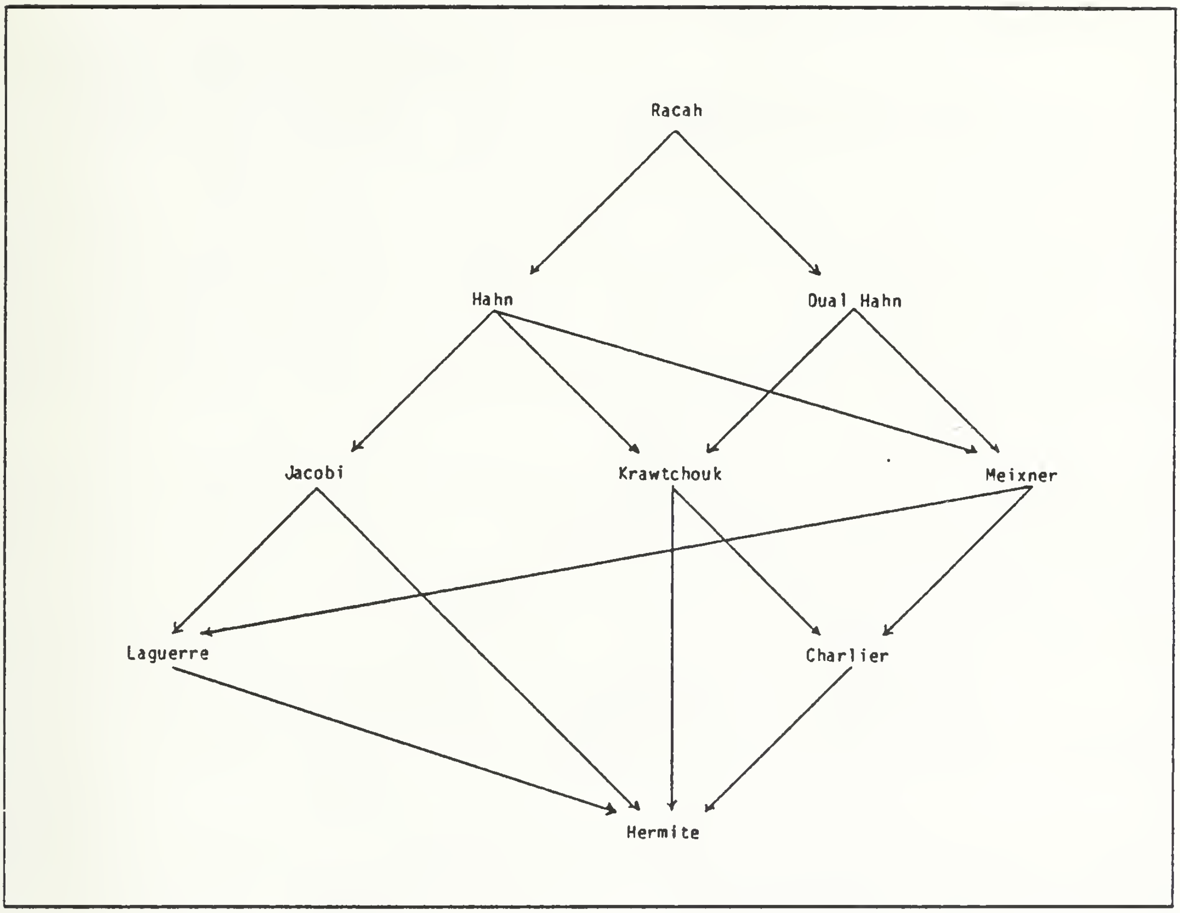

Figure 5. Hierarchy of Limiting Cases

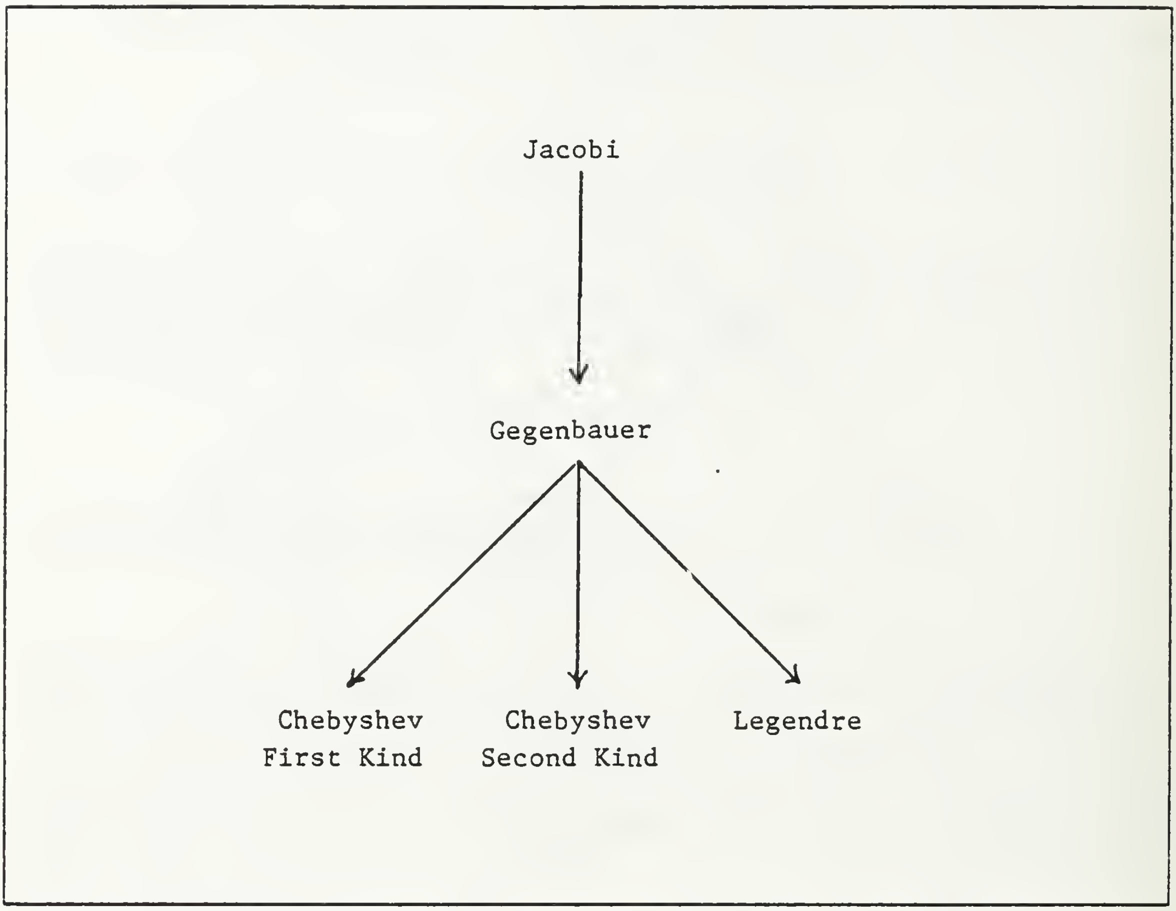

Figure 6. Hierarchy of Special Cases

Table 2. TABLE OF CLASSES

Class	Symbol	Hypergeometric Series
Racah	$R_n(\lambda(x);\alpha,\ \beta,\gamma,\delta)$	${}_4F_3\left[\begin{matrix}-n, n+\alpha+\beta+1, & -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix};1\right]$
Dual Hahn	$R_n(\lambda(x);\gamma,\delta,\ N)$	${}_3F_2\left[\begin{matrix}-n, & -x, & x+\alpha+\beta+1\\ & \gamma+1, & -N\end{matrix};1\right]$
Hahn	$Q_n(x;\alpha,\ \beta,\ N)$	${}_3F_2\left[\begin{matrix}-x, & -n, & n+\alpha+\beta+1\\ & \alpha+1, & -N\end{matrix};1\right]$
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$\dfrac{(\alpha+1)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n, & n+\alpha+\beta+1\\ & \alpha+1\end{matrix};\dfrac{1-x}{2}\right]$
Krawtchouk	$K_n(x;p,\ N)$	${}_2F_1\left[\begin{matrix}-n, & -x\\ & -N\end{matrix};\dfrac{1}{p}\right]$
Meixner	$M_n(x;\beta,c)$	${}_2F_1\left[\begin{matrix}-n, & -x\\ & \beta\end{matrix};1-\dfrac{1}{c}\right]$
Laguerre	$L_n^{(\alpha)}(x)$	$\dfrac{(\alpha+1)_n}{n!}\ {}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix};x\right]$
Charlier	$C_n(x;\alpha)$	${}_2F_0\left[\begin{matrix}-n, & -x\\ & -\end{matrix};-\dfrac{1}{\alpha}\right]$
Hermite	$H_n(x)$	$(2x)^n\ {}_2F_0\left[\begin{matrix}-n/2, & (-n+1)/2\\ & -\end{matrix};-\dfrac{1}{x^2}\right]$

Table 3. TABLE OF CLASSES (CONTINUED)

Gegenbauer	$C_n^{(\lambda)}(x)$	$\frac{(2\lambda)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ n+2\lambda \\ \lambda+1/2\end{matrix};\ \frac{1-x}{2}\right]$
Chebyshev, First Kind	$T_n(x)$	${}_2F_1\left[\begin{matrix}-n,\ n \\ 1/2\end{matrix};\ \frac{1-x}{2}\right]$
Chebyshev, Second Kind	$U_n(x)$	$(n+1)\,{}_2F_1\left[\begin{matrix}-n,\ n+1 \\ 3/2\end{matrix};\ \frac{1-x}{2}\right]$
Legendre	$P_n(x)$	${}_2F_1\left[\begin{matrix}-n,\ n+1 \\ 1\end{matrix};\ \frac{1-x}{2}\right]$

Table 4. RACAH POLYNOMIALS

Symbol: $R_n(\lambda(x);\alpha,\ \beta,\gamma,\delta)$

Interval: For N a positive integer, $x = 0, 1, \ldots, N$.

Weight:

$$w(x;\alpha,\ \beta,\gamma,\delta) = \frac{(\gamma+\delta+1)_x\,((\gamma+\delta+3)/2)_x\,(\alpha+1)_x\,(\beta+\delta+1)_x\,(\gamma+1)_x}{(1)_x\,((\gamma+\delta+1)/2)_x\,(\gamma+\delta-\alpha+1)_x\,(\gamma-\beta+1)_x\,(\delta+1)_x}\ .$$

Norm:

$$\sum_{x=0}^{N}[\,R_n(\lambda(x);\alpha,\ \beta,\gamma,\delta)\,]^2\,w(x;\alpha,\ \beta,\gamma,\delta)$$

$$= M\ \frac{n!\,(n+\alpha+\beta+1)_n\,(\beta+1)_n\,(\alpha-\delta+1)_n\,(\alpha+\beta-\gamma+1)_n}{(\alpha+\beta+2)_{2n}\,(\alpha+1)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n}$$

where if

$\alpha+1 = -N$,

$$M = \frac{(\gamma+\delta+2)_N\,(-\beta)_N}{(\gamma-\beta+1)_N\,(\delta+1)_N}\ ,$$

or if

$\beta+\delta+1 = -N$,

$$M = \frac{(\gamma+\delta+2)_N\,(\delta-\alpha)_N}{(\gamma+\delta-\alpha+1)_N\,(\delta+1)_N}\ ,$$

or if

$\gamma+1 = -N$,

$$M = \frac{(-\delta)_N\,(\alpha+\beta+2)_N}{(\alpha-\delta+1)_N\,(\beta+1)_N}\ .$$

Table 5. RACAH POLYNOMIALS (CONTINUED)

Hypergeometric Series:

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left[\begin{matrix}-n,\ n+\alpha+\beta+1,\ -x,\ x+\gamma+\delta+1\\ \alpha+1,\ \beta+\delta+1,\ \gamma+1\end{matrix};1\right]$$

where $\lambda(x) = x(x+\gamma+\delta+1)$, $n = 0, 1, \dots, N$,
and one of $\alpha+1$, $\beta+\delta+1$, or $\gamma+1$ equals $-N$.

Recurrence Relation:

$$\begin{aligned}\lambda(x)\, R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}& a(n)\left[R_{n+1}(\lambda(x);\alpha,\beta,\gamma,\delta) - R_n(\lambda(x);\alpha,\beta,\gamma,\delta)\right]\\ &-c(n)\left[R_n(\lambda(x);\alpha,\beta,\gamma,\delta) - R_{n-1}(\lambda(x);\alpha,\beta,\gamma,\delta)\right]\end{aligned}$$

where

$$a(n) = \frac{(n+\alpha+\beta+1)\,(n+\alpha+1)\,(n+\beta+\delta+1)\,(n+\gamma+1)}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta+2)}$$

$$c(n) = \frac{n\,(n+\beta)\,(n+\alpha+\beta-\gamma)\,(n+\alpha-\delta)}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta)}$$

Difference Equation:

$$\begin{aligned}\lambda(n)\, R_x(\lambda(n);\alpha,\beta,\gamma,\delta) = {}& A(x)\left[R_{x+1}(\lambda(n);\alpha,\beta,\gamma,\delta) - R_x(\lambda(n);\alpha,\beta,\gamma,\delta)\right]\\ &-C(x)\left[R_x(\lambda(n);\alpha,\beta,\gamma,\delta) - R_{x-1}(\lambda(n);\alpha,\beta,\gamma,\delta)\right]\end{aligned}$$

where

$$A(x) = \frac{(x+\gamma+\delta+1)\,(x+\gamma+1)\,(x+\delta+\beta+1)\,(x+\alpha+1)}{(2x+\gamma+\delta+1)\,(2x+\gamma+\delta+2)}$$

$$C(x) = \frac{x\,(x+\delta)\,(x+\gamma+\delta-\alpha)\,(x+\gamma-\beta)}{(2x+\gamma+\delta+1)\,(2x+\gamma+\delta)}$$

Table 6. DUAL HAHN POLYNOMIALS

Symbol: $R_n(\lambda(x)\,;\gamma,\delta,\ N)$

Interval: For N a positive integer, $x = 0, 1, \ldots, N$.

Weight:

$$w(x\,;\gamma,\delta,\ N) = \frac{(\gamma+\delta+1)_x\,((\gamma+\delta+3)/2)_x\,(\gamma+1)_x\,(-N)_x\,(-1)^x}{(1)_x\,((\gamma+\delta+1)/2)_x\,(\delta+1)_x\,(\gamma+\delta+N+2)_x}$$

Norm:

$$\sum_{x=0}^{N}[\,R_n(\lambda(x)\,;\gamma,\delta,\ N)\,]^2\,w(x\,;\gamma,\delta,\ N) = \left[\binom{n+\gamma}{n}\binom{N-n+\delta}{N-n}\right]^{-1}$$

Hypergeometric Series:

$$R_n(\lambda(x)\,;\gamma,\delta,\ N) = {}_3F_2\left[\begin{matrix}-n,\ -x,\ x+\gamma+\delta+1\\ \gamma+1,\ -N\end{matrix}\,;1\right]$$

where $\lambda(x) = x\,(x+\gamma+\delta+1)$

Recurrence Relation:

$$\begin{aligned}-\lambda(x)\,R_n(\lambda(x)\,;\gamma,\delta,\ N) = {}& B(n)\,[\,R_{n+1}(\lambda(x)\,;\gamma,\delta,\ N)-R_n(\lambda(x)\,;\gamma,\delta,\ N)\,]\\ &-D(n)\,[\,R_n(\lambda(x)\,;\gamma,\delta,\ N)-R_{n-1}(\lambda(x)\,;\gamma,\delta,\ N)\,]\end{aligned}$$

where

$$B(n) = (N-n)\,(\gamma+1+n)$$
$$D(n) = n(N+1+\delta-n)$$

Difference Equation:

$$\begin{aligned}-nR_n(\lambda(x)\,;\gamma,\delta,\ N) = {}& b(x)\,[\,R_n(\lambda(x+1)\,;\gamma,\delta,\ N)-R_n(\lambda(x)\,;\gamma,\delta,\ N)\,]\\ &-d(x)\,[\,R_n(\lambda(x)\,;\gamma,\delta,\ N)-R_n(\lambda(x-1)\,;\gamma,\delta,\ N)\,]\end{aligned}$$

where

$$b(x) = \frac{(x+\gamma+\delta+1)\,(x+\gamma+1)\,(N-x)}{(2x+\gamma+\delta+1)\,(2x+\gamma+\delta+2)}$$

$$d(x) = \frac{x\,(x+\delta)\,(x+\gamma+\delta+N+1)}{(2x+\gamma+\delta)\,(2x+\gamma+\delta+1)}$$

Table 7. HAHN POLYNOMIALS

Symbol: $Q_n(x\,;\alpha,\ \beta,\ N)$

Interval: For N a positive integer, $x=0, 1, \ldots, N$.

Weight:

$$w(x\,;\alpha,\ \beta,\ N)=\binom{x+\alpha}{x}\binom{N-x+\beta}{N-x}=\frac{(\beta+1)_N}{(1)_N}\ \frac{(\alpha+1)_x\,(-N)_x}{(1)_x\,(-N-\beta)_x}$$

Norm:

$$\sum_{x=0}^{\infty}[\,Q_n(x\,;\alpha,\ \beta,\ N)\,]^2\,w(x\,;\alpha,\ \beta,\ N)$$

$$=\frac{\alpha+\beta+1}{2n+\alpha+\beta+1}\ \frac{\binom{N+\alpha+\beta+1}{N}\binom{N+\alpha+\beta+1+n}{n}}{\binom{N}{n}}\times$$

$$\frac{\Gamma(\alpha+1)\,\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\ \frac{\Gamma(n+\beta+1)\,\Gamma(n+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+\beta+1)}$$

Hypergeometric Series: For $\alpha,\ \beta>-1$ and $n=0, 1, \ldots, N$,

$$Q_n(x\,;\alpha,\ \beta,\ N)={}_3F_2\left[\begin{matrix}-n,\ n+\alpha+\beta+1,\ -x\\ \alpha+1,\ -N\end{matrix}\,;\,1\right].$$

Table 8. HAHN POLYNOMIALS (CONTINUED)

Recurrence Relation:

$$-x\,Q_n(x\,;\alpha,\ \beta,\ N) = b(n)\left[\,Q_{n+1}(x\,;\alpha,\ \beta,\ N) - Q_n(x\,;\alpha,\ \beta,\ N)\,\right] \\ -d(n)\left[\,Q_n(x\,;\alpha,\ \beta,\ N) - Q_{n-1}(x\,;\alpha,\ \beta,\ N)\,\right]$$

where

$$b(n) = \frac{(n+\alpha+\beta+1)\,(n+\alpha+1)\,(N-n)}{(2n+\alpha+\beta+1)\,(2n+\alpha+\beta+2)}$$

$$d(n) = \frac{n\,(n+\beta)\,(n+\alpha+\beta+N+1)}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+1)}$$

Difference Equation:

$$-\lambda(n)\,Q_n(x\,;\alpha,\ \beta,\ N) = B(x)\left[\,Q_n(x+1\,;\alpha,\ \beta,\ N) - Q_n(x\,;\alpha,\ \beta,\ N)\,\right] \\ -D(x)\left[\,Q_n(x\,;\alpha,\ \beta,\ N) - Q_n(x-1\,;\alpha,\ \beta,\ N)\,\right]$$

where

$$B(x) = (N-x)\,(\alpha+1+x)$$

$$D(x) = x\,(N+1+\beta-x)$$

$$\lambda(n) = n\,(n+\alpha+\beta+1)$$

Rodrigues' Formula:

$$\binom{x+\alpha}{x}\binom{N-x+\beta}{N-x}\binom{N}{n}Q_n(x\,;\alpha,\ \beta,\ N) \\ = \binom{n+\beta}{n}\Delta^n\left[\binom{x+\alpha}{\alpha+n}\binom{N-x+\beta+n}{\beta+n}\right]$$

Table 9. JACOBI POLYNOMIALS

Symbol: $P_n^{(\alpha,\beta)}(x)$

Interval: $[-1,1]$

Weight: $(1-x)^\alpha (1+x)^\beta$

Standardization: $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \dfrac{(\alpha+1)_n}{n!}$

Norm:

$$\int_{-1}^{1} \left[P_n^{(\alpha,\beta)}(x)\right]^2 (1-x)^\alpha (1+x)^\beta \, dx = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}$$

Hypergeometric Series: $P_n^{(\alpha,\beta)}(x) = \dfrac{(\alpha+1)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n\,,\ n+\alpha+\beta+1\\ \alpha+1\end{matrix};\ \dfrac{1-x}{2}\right]$

Recurrence Relation: $2\,(n+1)\,(n+\alpha+\beta+1)\,(2n+\alpha+\beta)\,P_{n+1}^{(\alpha,\beta)}(x)$

$$= (2n+\alpha+\beta+1)\left[\,(2n+\alpha+\beta)\,(2n+\alpha+\beta+2)\,x+\alpha^2-\beta^2\,\right] P_n^{(\alpha,\beta)}(x)$$

$$-2\,(n+\alpha)\,(n+\beta)\,(2n+\alpha+\beta+2)\,P_{n-1}^{(\alpha,\beta)}(x)$$

Differential Equation:

$$\left(1-x^2\right)y'' + \left[\,\beta-\alpha-(\alpha+\beta+2)\,x\right]y' + n\,(n+\alpha+\beta+1)\,y = 0\,,\quad y = P_n^{(\alpha,\beta)}(x)$$

Rodrigues' Formula:

$$2^n\,n!\,P_n^{(\alpha,\beta)}(x) = (-1)^n\,(1-x)^{-\alpha}\,(1+x)^{-\beta}\,\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$$

Generating Function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,t^n = \frac{2^{\alpha+\beta}}{\sqrt{\left(1-2xt+t^2\right)}}\left(1-t+\sqrt{1-2xt+t^2}\right)^{-\alpha}\left(1+t+\sqrt{1-2xt+t^2}\right)^{-\beta}$$

Explicit Expression: $P_n^{(\alpha,\beta)}(x) = 2^{-n}\sum\limits_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k$

Table 10. LAGUERRE POLYNOMIALS

Symbol: $L_n^{(\alpha)}(x)$

Interval: $[0, \infty)$

Weight: $x^\alpha e^{-x}$, $\alpha > -1$

Standardization: $L_n^{(\alpha)}(1) = \dfrac{(-1)^n}{n!} x^n + \ldots$

Norm:

$$\int_0^\infty [L_n^{(\alpha)}(x)]^2 x^\alpha e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}$$

Hypergeometric Series:

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} {}_1F_1\left[\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right]$$

Recurrence Relation:

$$(n+1)\, L_{n+1}^{(\alpha)}(x) = [\, (2n+\alpha+1) - x\,]\, L_n^{(\alpha)}(x) - (n+\alpha)\, L_{n-1}^{(\alpha)}(x)$$

Differential Equation: $x y'' + (\alpha+1-x) y' + n y = 0$, $y = L_n^{(\alpha)}(x)$

Rodrigues' Formula:

$$L_n^{(\alpha)}(x) = \frac{1}{n!\, x^\alpha e^{-x}} \frac{d^n}{dx^n} [\, x^{n+\alpha} e^{-x}\,]$$

Generating Function:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right)$$

Explicit Expression:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{\infty} (-1)^k \binom{n+\alpha}{n-k} \frac{1}{k!} x^k$$

Table 11. HERMITE POLYNOMIALS

Symbol: $H_n(x)$

Interval: $(-\infty, \infty)$

Weight: e^{-x^2}

Standardization: $H_n(1) = 2^n x^n + \ldots$

Norm:

$$\int_{-\infty}^{\infty} [\, H_n(x) \,]^2 \, e^{-x^2} \, dx = \sqrt{\pi} \; 2^n \, n!$$

Hypergeometric Series:

$$H_n(x) = (2x)^n \, {}_2F_0\left[\begin{matrix} -n/2 \,, & (-n+1)/2 \\ - & \end{matrix} ; -\frac{1}{x^2} \right]$$

Recurrence Relation:

$$H_{n+1}(x) = 2x \, H_n(x) - 2n \, H_{n-1}(x)$$

Differential Equation:

$$y'' - 2\,x\,y' + 2\,n\,y = 0 \,, \quad y = H_n(x)$$

Rodrigues' Formula:

$$H_n(x) = (-1)^n \, e^{-x^2} \, \frac{d^n}{dx^n} \left[\, e^{-x^2} \, \right]$$

Generating Function:

$$\sum_{n=0}^{\infty} \frac{H_n(x) \, t^n}{n!} = e^{2xt - t^2}$$

Explicit Expression:

$$H_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \, \frac{(2x)^{n-2k}}{k! \, (n-2k)!}$$

Table 12. GEGENBAUER (ULTRASPHERICAL) POLYNOMIALS

Symbol: $C_n^{(\lambda)}(x)$ (or $P_n^{(\lambda)}(x)$), $\lambda > -1/2$

Interval: $[-1,1]$

Weight: $(1-x^2)^{\lambda-1/2}$

Standardization:

$$\lim_{\lambda\to 0} \frac{C_n^{(\lambda)}(x)}{C_n^{(\lambda)}(1)} = T_n(x)\ ,\ \ n = 0, 1, 2, \ldots$$

Norm:

$$\int_{-1}^{1} [C_n^{(\lambda)}(x)]^2 \left(1-x^2\right)^{\lambda-1/2} dx = \frac{2^{2\lambda-1}\,[\,\Gamma(\,n+\lambda+1/2)\,]^2}{(n+\lambda)\; n!\; \Gamma(n+2\lambda)}$$

Hypergeometric Series:

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n\ ,\ n+2\lambda \\ \lambda+1/2 \end{matrix}\ ;\ \frac{1-x}{2}\right]$$

Recurrence Relation:

$$(n+1)\, C_{n+1}^{(\lambda)}(x) = 2\,(n+\lambda)\, x\, C_n^{(\lambda)}(x) - (n+2\lambda-1)\, C_{n-1}^{(\lambda)}(x)$$

Differential Equation:

$$\left(1-x^2\right) y'' - (2\lambda+1)\, x\, y' + n\,(n+2\lambda)\, y = 0\ ,\ \ y = C_n^{(\lambda)}(x)$$

Rodrigues' Formula: $C_0^{(\lambda)}(x) = 1$, $C_1^{(\lambda)}(x) = 2\lambda x$,

$$2^n\, n!\, (\lambda+1/2)_n \left(1-x^2\right)^{\lambda-1/2} C_n^{(\lambda)}(x) = (-1)^n\, (2\lambda)_n\, \frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n+\lambda-1/2}\right]$$

Generating Function:

$$\sum_{n=0}^{\infty} C_n^{(\lambda)}(x)\, t^n = \left(1-2xt+t^2\right)^{-\lambda}$$

Table 13. CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

Symbol: $T_n(x)$

Interval: $[-1,1]$

Weight: $(1-x^2)^{-1/2}$

Standardization: $T_n(1)=1$

Norm:

$$\int_{-1}^{1}[T_n(x)]^2(1-x^2)^{-1/2}dx=\begin{cases}\pi/2, & n\neq 0\\ \pi, & n=0\end{cases}$$

Hypergeometric Series:

$$T_n(x)={}_2F_1\left[\begin{matrix}-n\,,\,n\\ 1/2\end{matrix}\,;\,\frac{1-x}{2}\right]$$

Recurrence Relation: $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$

Differential Equation: $(1-x^2)\,y''-xy'+n^2y=0\,,\quad y=T_n(x)$

Rodrigues' Formula:

$$T_n(x)=\frac{(-1)^n(1-x^2)^{-1/2}\sqrt{\pi}}{2^{n+1}\,\Gamma(n+1/2)}\frac{d^n}{dx^n}\left[(1-x^2)^{n-1/2}\right]$$

Generating Function:

$$\sum_{n=0}^{\infty}T_n(x)\,t^n=\frac{1-xt}{1-2xt+t^2}\;,\quad -1<x<1\,,\quad |\,t\,|<1$$

Explicit Expression:

$$T_n(x)=\cos(n\theta)\quad\text{with}\quad x=\cos\theta\,,\;0\leq\theta\leq\pi$$

$$T_n(x)=\frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}$$

Table 14. CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

Symbol: $U_n(x)$

Interval: $[-1,1]$

Weight: $(1-x^2)^{1/2}$

Standardization: $U_n(1) = n+1$

Norm:

$$\int_{-1}^{1} [U_n(x)]^2 \left(1-x^2\right)^{1/2} dx = \frac{\pi}{2}$$

Hypergeometric Series:

$$U_n(x) = (n+1)\,{}_2F_1\left[\begin{matrix} -n,\ n+1 \\ 3/2 \end{matrix};\ \frac{1-x}{2}\right]$$

Recurrence Relation: $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$

Differential Equation: $(1-x^2)\,y'' - 3xy' + n\,(n+2)\,y = 0\,,\quad y = U_n(x)$

Rodrigues' Formula:

$$U_n(x) = \frac{(-1)^n\,(n+1)\,\sqrt{\pi}}{\left(1-x^2\right)^{1/2}\,2^{n+1}\,\Gamma(n+3/2)}\ \frac{d^n}{dx^n}\left[\left(1-x^2\right)^{n+1/2}\right]$$

Generating Function:

$$\sum_{n=0}^{\infty} U_n(x)\,t^n = \frac{1}{1-2xt+t^2}\,,\quad -1 < x < 1\,,\quad |\,t\,| < 1$$

Explicit Expression:

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \quad \text{with} \quad x = \cos\theta\,,\ 0 \le \theta \le \pi$$

$$U_n(x) = \sum_{k=0}^{[n/2]} (-1)^k\,\frac{(n-k)!}{k!\,(n-2k)!}\,(2x)^{n-2k}$$

Table 15. LEGENDRE (SPHERICAL) POLYNOMIALS

Symbol: $P_n(x)$

Interval: $[-1,1]$

Weight: 1

Standardization: $P_n(1) = 1$

Norm: $\int_{-1}^{1} [P_n(x)]^2 dx = \dfrac{2}{2n+1}$

Hypergeometric Series: $P_n(x) = {}_2F_1\left[\begin{matrix} -n, n+1 \\ 1 \end{matrix} ; \dfrac{1-x}{2}\right]$

Recurrence Relation: $(n+1)\, P_{n+1}(x) = (2n+1)\, x\, P_n(x) - n\, P_{n-1}(x)$

Differential Equation: $(1-x^2)\, y'' - 2xy' + n\,(n+1)\, y = 0\,, \quad y = P_n(x)$

Rodrigues' Formula: $P_n(x) = \dfrac{(-1)^n}{2^n\, n!} \dfrac{d^n}{dx^n} [\,(1-x^2)^n]$

Generating Function:

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \left(1-2xt+t^2\right)^{-1/2}, \quad -1 < x < 1\,, \quad |\,t\,| < 1$$

Explicit Expression: $P_n(x) = \dfrac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}$

V. APPLICATIONS

Orthogonal polynomials and special functions in general have been studied extensively in mathematics and other fields since the eighteenth century. Presented below are a few of the traditional applications of selected classes of orthogonal polynomials. Our first few applications come from numerical analysis.

A. ECONOMIZATION OF POWER SERIES

Economization of power series is a technique used to reduce the degree of a polynomial approximation to a given function.

The *maximum norm* (or L^∞ – norm) for a continuous function on a compact interval $[a,b]$ is defined as

$$\|f\|_\infty = \max_{x \in [a,b]} |f(x)| .$$

This norm is not induced by an inner product, but nevertheless has many uses in applied mathematics, including numerical analysis.

The *minimax property* of the Chebyshev polynomials states that of all n^{th} degree monic polynomials (i.e., leading coefficient 1), $2^{1-n}T_n(x)$ has the smallest maximum norm on $[-1,1]$ [Ref. 23: p. 106]. The justification for this statement is deferred until Section C. Hence the best approximation in the maximum norm to the function x^n on $[-1,1]$ by a function of lower degree is $f_n(x) = x^n - 2^{1-n}T_n(x)$. So, given a function and a polynomial approximation to that function (e.g., from a Taylor series expansion), successively replace the highest powers x^n with $f_n(x)$ to obtain a polynomial approximation of lower degree [Ref. 23: p. 125].

Example 1: Let $f(x) = \sin x$. The Maclaurin series for this function is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

and this series is convergent for $x \in \mathbb{R}$. If truncated after the x^5 term, the polynomial approximation for the function is

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

with a maximum error of 0.0002 for $x \in [-1,1]$. We use the fifth degree Chebyshev polynomial $T_5(x) = 16x^5 - 20x^3 + 5x$ to obtain

$$\frac{x^5}{5!} \approx \frac{5x^3}{480} - \frac{5x}{1920}$$

with an error not exceeding

$$\frac{\max |\ T_5(x)\ |}{1920} = \frac{1}{1920} \approx 0.00052$$

in $[-1,1]$. Thus the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \left(\frac{5x^3}{480} - \frac{5x}{1920}\right) = \frac{1915}{1920}x - \frac{75}{480}x^3$$

has an error whose magnitude in $[-1,1]$ does not exceed

$$0.00052 + 0.0002 = 0.00072.$$

Compare this with the maximum error of 0.00833 for the Maclaurin series which is truncated after the x^3 term. For a cubic polynomial approximation of $\sin x$, the "economized" polynomial has a maximum error that is significantly smaller (less than one tenth) than that of the truncated Maclaurin series.

The next three applications illustrate the usefulness of the *zeros* of orthogonal polynomials. The first two come from numerical analysis, the third from a problem in electrostatics. We begin with a preliminary discussion of a fundamental technique from numerical analysis.

B. POLYNOMIAL INTERPOLATION

Polynomial interpolation is a method of approximating a given function with a polynomial that matches (*interpolates*) the function at specified points (called *nodes* or *abscissae)* $x_1, \ldots, x_n$ [Ref. 24: p. 497]. Given a function $f(x)$ and n distinct nodes in a compact interval $[a,b]$, there is a unique polynomial of degree $(n-1)$ that passes through the points $(x_i, f(x_i))$, $1 \leq i \leq n$.

For each $i = 1, 2, \ldots, n$, define a polynomial of degree $(n-1)$ by

$$\pi_{i,n}(x) = \prod_{\substack{k=1 \\ k\neq i}}^{n} \left(\frac{x-x_k}{x_i-x_k} \right).$$

Clearly, for each $k = 1, 2, \ldots, n$, $\pi_{i,n}(x_k) = 0$ if $k \neq i$ and $\pi_{i,n}(x_i) = 1$, that is, $\pi_{i,n}(x_k) = \delta_{i,k}$. (We may thus equivalently express $\pi_{i,n}(x) = l_n(x)/((x-x_i)\, l_n'(x_i))$, where $l_n(x) = \prod_{k=1}^{n} (x-x_k)$ is the unique monic polynomial of degree n that vanishes simply at each node x_k, $k = 1, 2, \ldots, n$.)

The *Lagrange interpolating polynomial* is now given by

$$L_n^f(x) = \sum_{i=1}^{n} f(x_i)\, \pi_{i,n}(x).$$

Clearly, $L_n^f(x_k) = f(x_k)$ for $k = 1, 2, \ldots, n$, and uniqueness of the degree $(n-1)$ interpolating polynomial is guaranteed by the Fundamental Theorem of Algebra.

C. OPTIMAL NODES

In this section we address the issue of estimating the maximum size of the interpolating error $\| f - L_n^f \|_\infty$. Assuming f is suitably differentiable in $[a,b]$, it can be shown [Ref. 25: p. 188] that there exists a value K_n (which depends on $f^{(n)}$ in $[a,b]$) such that for any x in $[a,b]$,

$$\left| f(x) - L_n^f(x) \right| \leq \frac{K_n}{n!} \; | \, l_n(x) \, | ,$$

where $l_n(x) = \prod_{i=1}^{n} (x - x_i)$. (Note again that $l_n(x)$ is a *monic* polynomial of degree n which vanishes at the nodes.) This implies that $\| f - L_n^f \|_\infty$ is minimized by making the optimal choice of nodes $x_1, \ldots, x_n$ in $[a,b]$ which minimizes $\| \, l_n \, \|_\infty$. Surprisingly perhaps, this happens precisely at the zeros of the Chebyshev polynomials $T_n(x)$, scaled to the interval $[a,b]$. We sketch the reasons below.

For simplicity, we take our interval of interest to be $[-1,1]$ instead of $[a,b]$ without loss of generality. The transformation

$$x = \left(\frac{b-a}{2} \right) t + \left(\frac{b+a}{2} \right),$$

or equivalently,

$$t = 2\left(\frac{x-a}{b-a}\right) - 1,$$

is a one-to-one continuous mapping between the intervals $[-1,1]$ and $[a,b]$.

Now let x_i be such that $T_n(x_i) = 0$, $i = 1, 2, \dots, n$ (see Chapter I, Section A.3), i.e., let $l_n(x) = \frac{1}{2^{n-1}} T_n(x)$. From the recurrence relation (Equation (4) in Chapter I), $T_n(x)$ has a leading coefficient of 2^{n-1} ; hence this $l_n(x)$ is monic. Moreover, from the definition $T_n(\cos\theta) = \cos n\theta$, it follows that

(1) $$l_n(y_i) = (-1)^i \frac{1}{2^{n-1}}$$

where $y_i = y_{i,n} = \frac{\cos i\pi}{n}$, $i = 0, 1, \dots, n$. Now suppose $m_n(x)$ is another monic polynomial of degree n such that

(2) $$\| m_n \|_\infty < \| l_n \|_\infty = \frac{1}{2^{n-1}} .$$

Combining (1) and (2) we see that we must have for $i = 0, 1, \dots, n$,

(3) $$\begin{aligned} m_n(y_i) < l_n(y_i) = \frac{1}{2^{n-1}} \quad &\text{if } i \text{ is even, and} \\ m_n(y_i) > l_n(y_i) = \frac{-1}{2^{n-1}} \quad &\text{if } i \text{ is odd.} \end{aligned}$$

Thus, the polynomial $p_n(x) = m_n(x) - l_n(x)$ has degree at most $(n-1)$ (since it is a difference of two *monic* polynomials of degree n) with at least n zeros, one in each interval (y_i, y_{i+1}), $0 \le i \le n-1$, by (3). This contradicts the Fundamental Theorem of Algebra, and so no such polynomial $m_n(x)$ satisfying (2) exists. Hence the choice of nodes x_i determined by $l_n(x) = \frac{1}{2^{n-1}} T_n(x)$ minimizes $\| l_n \|_\infty = \max_{-1 \le x \le 1} | l_n(x) |$ over all possible monic polynomials $l_n(x)$, and therefore over all possible choices of interpolating nodes x_i.

It should be emphasized that this *Chebyshev interpolation* allows an *a priori* error bound for all x, but is not always best possible for every x using other interpolation schemes. (For example, if equally spaced nodes $x_1, \dots, x_n$ are used, then trivially, the error $f(x_i) - L_n(x_i) = 0$, even if $x = x_i$ is not a Chebyshev zero.) Chebyshev zeros are optimal when one has freedom in the choice of nodes. For details regarding the practicality of Chebyshev interpolation and some asymptotic results, see [Ref. 24].

D. GAUSSIAN QUADRATURE

Quadrature formulas are used in that area of numerical analysis concerned with the approximate integration of a function $f(x)$ against a weight function $w(x) > 0$ on an interval (a,b), when the explicit evaluation is intractable.

An *interpolatory quadrature* is such a rule that uses interpolating polynomials, such as Lagrange polynomials:

$$\int_a^b f(x)\, w(x)\, dx \approx \int_a^b L_n^f(x)\, w(x)\, dx,$$

where $L_n^f(x) = \sum_{i=1}^{n} f(x_i)\, \pi_i(x)$, as described above. This can be rewritten as

$$(1) \qquad \int_a^b f(x)\, w(x)\, dx \approx \sum_{i=1}^{n} f(x_i)\, w_i,$$

where the weights w_i are given by

$$(2) \qquad w_i = \int_a^b \pi_{i,n}(x)\, w(x)\, dx.$$

Thus, interpolatory quadrature is basically a weighted sum of the function values $f(x_i)$ at the nodes x_i, $i = 1, 2, \ldots, n$, as are numerical integration recipes such as Simpson's Rule and the Trapezoidal Rule.

For *specified* nodes $x_1, \ldots, x_n \in [a,b]$, the n weights $w_1, \ldots, w_n$ computed in (2) for the quadrature (1) will be *exact* for polynomials $f \in P_{n-1}[a,b]$, but we can do better. [Ref. 25: p. 236]

In *Gaussian quadrature,* we ask for the location of the n nodes $x_1, \ldots, x_n$ *as well as* the n weights $w_1, \ldots, w_n$ in order for the quadrature rule (1) to be exact for polynomials $f \in P_{2n-1}[a,b]$. At first glance, this seems to be an extremely complicated computational problem, but the solution falls out simply when the theory of orthogonal polynomials is applied.

Let $\{p_n(x)\}_{n=0}^{\infty}$ be the class of polynomials orthogonal with respect to the weight function $w(x)$ on $[a,b]$, say, by Gram-Schmidt, and let $L_n^f(x)$ be the Lagrange polynomial that interpolates $f \in P_{2n-1}[a,b]$ *at the zeros of* $p_n(x)$, so that $f(x) - L_n^f(x) \in P_{2n-1}[a,b]$ vanishes at the zeros also. Since the zeros of $p_n(x)$ are real, simple, and lie in (a,b), we have the property that

$$f(x) - L_n^f(x) = p_n(x)\, q_{n-1}(x),$$

where $q_{n-1} \in P_{n-1}[a,b]$. Hence, by orthogonality,

$$\int_a^b f(x)\, w(x)\, dx - \int_a^b L_n^f(x)\, w(x)\, dx = \int_a^b p_n(x)\, q_{n-1}(x)\, w(x)\, dx = 0.$$

We have thus shown that the resulting *n-point Gaussian quadrature rule* is exact for $f \in P_{2n-1}[a,b]$. Moreover, by a theorem of Stieltjes, if $f(x)$ is continuous on a finite interval $[a,b]$, then

$$\lim_{n\to\infty} \int_a^b L_n^f(x)\, w(x)\, dx = \int_a^b f(x)\, w(x)\, dx.$$

By applying the Christoffel-Darboux formula (Equation (9) of Chapter III with notation from Chapter III, Section B) and using (1), it is possible to derive an alternate expression for the weights (2):

$$w_i = -\frac{A_{n+1}}{h_n\, p_n{}'(x_i)\, p_{n+1}(x_i)}$$

which are referred to as the *Christoffel numbers.* [Ref. 8]

Thus Gauss-Jacobi, Gauss-Chebyshev, and Gauss-Legendre are the names given to Gaussian quadratures involving the weight functions and orthogonal polynomials from the Jacobi, Chebyshev, and Legendre classes, respectively.

E. ELECTROSTATICS

The zeros of the Jacobi polynomials play an interesting role in a problem of Stieltjes concerning electrostatic equilibrium. In this problem, fix "masses" of positive charge α and β at the points $x = 1$ and $x = -1$, respectively. Then place n point masses of positive unit charge in the interval $(-1,1)$ so that they are free to move. These interior masses are now subject to a "logarithmic potential", that is, a repelling force that is proportional to the logarithm of the distance separating them. The problem is to determine the distribution of the point masses x_i, $i = 1, 2, \ldots, n$, when the system is in equilibrium. Mathematically, this is equivalent to maximizing the force function

$$(3) \qquad F(x_1, \dots, x_n) = \alpha \sum_{i=1}^{n} \log(1-x_i) + \beta \sum_{i=1}^{n} \log(1+x_i) + \sum_{1 \le i < j \le n} \log |x_i - x_j| .$$

The logarithmic terms give the restrictions $x_1 \neq -1$, $x_n \neq 1$, and $x_i \neq x_j$ for $i \neq j$. This otherwise continuous function gives the equilibrium points by setting

$$(4) \qquad \frac{\partial F}{\partial x_i} = 0 \, , \ i = 1, 2, \dots, n$$

and solving for $x_1, \dots, x_n$. To solve this system of n nonlinear equations in n unknowns, Stieltjes introduced the polynomial

$$p_n(x) = \prod_{i=1}^{n} (x - x_i)$$

and reduced (4) to

$$\frac{1}{2} \frac{p_n''(x_i)}{p_n'(x_i)} + \frac{\beta}{1+x_i} - \frac{\alpha}{1-x_i} = 0 \, , \ i = 1, 2, \dots, n$$

which becomes

$$\left(1-x^2\right) p_n''(x_i) + [\, 2\beta - 2\alpha - (2\alpha + 2\beta)\, x_i \,]\, p_n'(x_i) = 0$$

for $i = 1, 2, \dots, n$.

Since the polynomial

$$\left(1 - x^2\right) p_n''(x) + [\, 2\beta - 2\alpha - (2\alpha + 2\beta)\, x \,]\, p_n'(x)$$

is of degree at most n and vanishes at $x = x_i$, $i = 1, 2, \dots, n$, it can be set equal to a scalar multiple $\lambda p_n(x)$ which also vanishes at these points. Hence

$$\left(1 - x^2\right) p_n''(x) + [\, 2\beta - 2\alpha - (2\alpha + 2\beta)\, x \,]\, p_n'(x) - \lambda p_n(x) = 0.$$

Attempting a power series solution to this second-order differential equation leads to the observation that *polynomial* (i.e., terminating) solutions exist if and only if

$$\lambda = -n(n + 2\alpha + 2\beta - 1).$$

Rearranging terms yields

$$\left(1-x^2\right)p_n''(x)+\left[\,(2\beta-1)-(2\alpha-1)-((2\alpha-1)+(2\beta-1)+2)\,x\,\right]p_n'(x)$$
$$+n(n+(2\alpha-1)+(2\beta-1)+1)\,p_n(x)=0$$

which is the differential equation for $P_n^{(2\alpha-1,\,2\beta-1)}(x)$. Thus the equilibrium positions of the unit charges occur at the zeros of this Jacobi polynomial. [Refs. 8, 13: p. 140]

The zeros of the Laguerre and Hermite polynomials can be developed as solutions to similar electrostatic equilibrium problems.

F. SPHERICAL HARMONICS

We investigate another application of orthogonal polynomials to problems in mathematical physics. In Cartesian coordinates (x, y, z), the *Laplacian operator* of a function $u(x, y, z)$ is defined as

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}.$$

When converted to spherical coordinates (r, θ, ϕ), this operator acting on a function $u(r, \theta, \phi)$ becomes

$$\nabla^2 u=\frac{1}{r^2}\left[\left(r^2 u_r\right)_r+\frac{1}{\sin\theta}\,(\sin\theta\, u_\theta)_\theta+\frac{1}{\sin^2\theta}\,u_{\theta\,\theta}\right] \tag{5}$$

where subscripting with r, θ, or ϕ denotes partial differentiation with respect to that variable. A function u is said to be *harmonic* in a region D if in that region it satisfies *Laplace's equation:*

$$\nabla^2 u=0.$$

In particular, if the boundary of D is the unit sphere centered at the origin, then using the method of separation of variables, the Legendre polynomials arise naturally as part of the solution.

In separation of variables, we assume that the solution will be of the form

$$u(r,\ \theta,\ \phi)=f(r)\,g(\theta)\,h(\phi). \tag{6}$$

Substituting (6) into (5) and dividing through by u, we obtain

$$\frac{1}{f}\left(r^2 f_r\right)_r + \frac{1}{g\,\sin\theta}\,(\sin\theta\, g_\theta)_\theta + \frac{1}{h\,\sin^2\theta}\, h_{\phi\phi} = 0. \tag{7}$$

The first term, depending only on r, must reduce to a constant which we write as $v(v+1)$. Substituting $v(v+1)$ into (7) and multiplying through by $\sin^2\theta$, we have

$$v(v+1)\,\sin^2\theta + \frac{\sin\theta}{g}\,(\sin\theta\, g_\theta)_\theta + \frac{1}{h}\, h_{\phi\phi} = 0.$$

Now we see that the third term, depending only on ϕ, must also reduce to a constant; call it $-m^2$. Substituting $-m^2$ for this term and simplifying, we obtain

$$\sin^2\theta\, g_{\theta\theta} + \sin\theta\,\cos\theta\, g_\theta + \left[v(v+1)\sin^2\theta - m^2\right] g = 0. \tag{8}$$

By the change of variable $x = \cos\theta$, (8) becomes via the Chain Rule

$$\left(1-x^2\right)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + \left[\, v(v+1) - \frac{m^2}{1-x^2}\,\right] g = 0.$$

For problems that are radially symmetric, the ϕ dependence can be removed by setting $m = 0$, leaving

$$\left(1-x^2\right)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + v(v+1)\, g = 0. \tag{9}$$

When v is a positive integer n, (9) is recognized as the differential equation for the Legendre polynomials. [Ref. 26: pp. 210-213]

An alternate approach to this problem (by the method of images) uses *Green's functions.* As motivation, consider the *Dirichlet problem for the unit circle* in the plane, which involves finding a harmonic function $u(r,\ \theta)$ in the unit disk that takes on prescribed function values $f(\theta)$ on the boundary $r = 1$. The solution is given by

$$u(r,\ \theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, P_r(\theta-\theta')\, d\theta',$$

where

$$P_r(\gamma) = \frac{1-r^2}{1-2r\cos\gamma + r^2}$$

is the so-called *Poisson kernel* for this problem.

Similarly, the solution to the *Dirichlet problem for the unit sphere* in $\mathbb{R}^3$

$$\begin{cases} \nabla^2 u = 0 \\ u = f(\theta, \phi) \ \text{ on } \ r = 1 \end{cases}$$

can be expressed as

$$u(r, \theta, \phi) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\theta', \phi')\, P_r(\gamma) \ \sin\theta' \, d\theta' \, d\phi'$$

where

(10)
$$P_r(\gamma) = \frac{1-r^2}{\left[\, 1-2r\cos\gamma+r^2 \,\right]^{3/2}}$$

and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi').$$

Writing (10) in terms of simpler functions

$$P_r(\gamma) = \frac{1}{\sqrt{1-2r\cos\gamma+r^2}} + 2r\,\frac{\partial}{\partial r}\left[\frac{1}{\sqrt{1-2r\cos\gamma+r^2}}\right],$$

we note the appearance of the generating function for the Legendre polynomials, with $x = \cos\gamma$. Therefore the Legendre polynomials are again part of the solution. [Ref. 27: pp. 87-89]

Laplace's equation can also be solved in a higher dimensional setting. Let $\mathrm{x} = (x_1, \dots, x_p) \in \mathbb{R}^p$. Note then that we may write $\mathrm{x} = r\xi$, where $r = \| \mathrm{x} \| = \sqrt{x_1^2 + \ \dots \ + x_p^2}$ and $\xi = \frac{\mathrm{x}}{r} = (\xi_1, \dots, \xi_p) \in \mathbb{R}^p$ is a unit vector. A polynomial $h_n(\mathrm{x})$ is said to be *homogeneous* of degree n if $h_n(\lambda\mathrm{x}) = \lambda^n\, h_n(\mathrm{x})$.

The Laplacian operator acting on a function $u(\mathrm{x})$ in p dimensions is defined as

$$\nabla_p^2 u = \sum_{i=1}^{p} \frac{\partial^2 u}{\partial x_i^2}.$$

As before, a harmonic function is one which satisfies Laplace's equation $\nabla_p^2 u = 0$. We now seek homogeneous harmonic polynomials of degree n in $\mathbf{x} \in \mathbb{R}^p$. It can be shown that there are exactly

$$N = N_{p,\,n} = \frac{2n+p-2}{n} \binom{n+p-3}{n-1}$$

linearly independent such solutions, and they can be characterized by Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda)}(\mathbf{t}_k)$, $k = 1, 2, \ldots, N$. The general solution is given by

$$h_n(\mathbf{x}) = h_n(r\xi) = r^n \, S_n(\xi),$$

where the *spherical harmonic*

$$S_n(\xi) = \sum_{k=1}^{N} A_{k,\,n,\,p} \, C_n^{\left(\frac{p-2}{2}\right)}((\xi,\, \eta_k)),$$

with η_k suitably chosen unit vectors. Note that if $p = 3$, then these reduce to the Legendre polynomials found earlier. [Ref. 6: pp. 168-183]

G. GENETICS MODELING

Karlin and McGregor gave an interesting application of the dual Hahn polynomials to a model in genetics. In this continuous time Markoff chain model, the dual Hahn polynomials $R_n(\,\lambda(x)\,;\gamma,\,\delta,\;N\,)$ arise in the transition probability function for the process.

The setting for the model assumes N gametes of type a or A and gives a random fertilization scheme. The population of either type of gamete is affected by both the fertilization process and a mutation process whereby a gamete resulting from a mating can mutate into the other type.

By considering the conditional probabilities for an increase in population size of both gamete types, a stochastic process is defined which is a classical birth and death process. The transition probability function for this last process is then cast in terms of the dual Hahn polynomials. The interested reader is referred to [Ref. 14] for the details.

VI. BASIC EXTENSIONS

A. BASIC HYPERGEOMETRIC SERIES

In this chapter, we extend the structure and some of the results in Chapter IV to a more general level. This extension is accomplished by introducing a new parameter called the *base* to the hypergeometric series. The base q was used by Heine in a series

$$(1) \qquad 1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}x+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}x^2+\ldots$$

where $c \neq 0, -1, -2, \ldots$ [Refs. 28,29]. This series converges absolutely for $|x| < 1$ when $|q| < 1$ by the Ratio test. Since

$$(2) \qquad \lim_{q\to 1}\frac{1-q^a}{1-q}=a,$$

we see that the series in (1) tends termwise to the ordinary hypergeometric series as $q \to 1$. Thus Heine's series is called the *basic hypergeometric series* or the *q-hypergeometric series*. [Ref. 12: p. 3]

The *q-shifted factorial* is the basic extension of the shifted factorial introduced in Chapter II and is defined

$$(3) \qquad (a\,;q)_n=\begin{cases}1, & n=0\\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & n=1,2,\ldots.\end{cases}$$

The ordinary shifted factorial is recovered by applying (2) and (3) in the limit

$$\lim_{q\to 1}\frac{(q^\alpha\,;q)_n}{(1-q)^n}=(\alpha)_n.$$

We also define

$$(4) \qquad (a\,;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k).$$

a form we will see in later results. Since the infinite product diverges when both $a \neq 0$ and $|q| \geq 1$, we will assume $|q| < 1$ whenever $(a\,;q)_\infty$ appears unless otherwise stated. [Ref. 12: p. 3]

Generalizing the basic hypergeometric series above, we define the ${}_r\phi_s$ *basic hypergeometric series* (or ${}_r\phi_s$ series)

$$(5) \qquad {}_r\phi_s\left[\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, x\right] = \sum_{n=0}^{\infty} \frac{(a_1\,;q)_n \cdots (a_r\,;q)_n}{(q\,;q)_n\,(b_1\,;q)_n \cdots (b_s\,;q)_n} \left((-1)^n q^{\binom{n}{2}}\right)^{1+s-r} x^n$$

where $\binom{n}{2} = n(n-1)/2$ and $q \neq 0$ when $r > s+1$. We require that the parameters $b_1, \dots, b_s$ be such that the denominator factors in each term of the series are nonzero. Since

$$(q^{-m}\,;q)_n = 0\,, \quad n = m+1, m+2, \dots\,,$$

a ${}_r\phi_s$ series terminates if one or more of the numerator parameters is of the form q^{-m} for $m = 0, 1, 2, \dots$ and $q \neq 0$. When $r = s+1$, the expression in (5) simplifies to

$${}_{s+1}\phi_s\left[\begin{matrix} a_1, \dots, a_{s+1} \\ b_1, \dots, b_s \end{matrix}; q, x\right] = \sum_{n=0}^{\infty} \frac{(a_1\,;q)_n \cdots (a_{s+1}\,;q)_n}{(q\,;q)_n\,(b_1\,;q)_n \cdots (b_s\,;q)_n}\, x^n.$$

Note that in a basic hypergeometric series $\sum c_n x^n$, the ratio c_{n+1}/c_n is a rational function of q^n. [Ref. 12: p. 4]

Using the $q-$shifted factorial (3) and (4), we can define basic extensions for many of the functions and formulas introduced in earlier chapters. We note that often there is more than one way to extend a result; examples will be given shortly.

The *q-gamma function* is defined by

$$(6) \qquad \Gamma_q(x) = \frac{(q\,;q)_\infty}{(q^x\,;q)_\infty}\,(1-q)^{1-x}\,, \quad 0 < q < 1.$$

Gosper showed that

$$(7) \qquad \lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x)$$

for $0 < q < 1$ [Ref. 30: p. 109]. The structure of the gamma function extends as well. For instance, the formula

$$\Gamma_q(x+1) = \left(\frac{1-q^x}{1-q} \right) \Gamma_q(x)$$

can be reduced using (6) to

$$\Gamma(x+1) = x\,\Gamma(x).$$

With (6) we can define the *q-beta function*

$$B_q(x,y) = \frac{\Gamma_q(x)\,\Gamma_q(y)}{\Gamma_q(x+y)}$$

which by (7) tends to $B(x,y)$ as $q \to 1^-$.

The *q-binomial coefficient* is defined for integers n and k by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q\,;q)_n}{(q\,;q)_k\,(q\,;q)_{n-k}},$$

where $k = 0, 1, \ldots, n$. For nonintegral α and β, we define

$$\begin{aligned} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q &= \frac{(q^{\beta+1}\,;q)_\infty\,(q^{\alpha-\beta+1}\,;q)_\infty}{(q\,;q)_\infty\,(q^{\alpha+1}\,;q)_\infty} \\ &= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\,\Gamma_q(\alpha-\beta+1)}. \end{aligned}$$

The *q-binomial theorem* is then

$$(ab\,;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k\,(a\,;q)_k\,(b\,;q)_{n-k}$$

where $n = 0, 1, 2, \ldots$. [Ref. 12: p. 20]

The next two expressions are basic extensions of the Chu-Vandermonde formula (Chapter III, Section F.1) and are both known as the *q-Chu-Vandermonde formula:*

$${}_2\phi_1 \left[\begin{matrix} q^{-n},\ b \\ c \end{matrix} ;q\,,\ \frac{cq^n}{b} \right] = \frac{(c/b\,;q)_n}{(c\,;q)_n}$$

$$
{}_2\phi_1\left[\begin{matrix} q^{-n},\ b \\ c \end{matrix};q\,,q\right]=\frac{(c/b\,;q)_n}{(c\,;q)_n}\,b^n
$$

[Ref. 12: p. 11]. These forms can be shown to be equivalent by reversing the order of summation.

Jackson introduced the general form of the *q-integral*

$$
\int_a^b f(t)\,d_qt=\int_0^b f(t)\,d_qt-\int_0^a f(t)\,d_qt
$$

where

$$
\int_0^a f(t)\,d_qt=a(1-q)\sum_{n=0}^{\infty}f(aq^n)\,q^n.
$$

The $q-$integral defines the measure d_qt which is a natural object for $q-$defined functions [Ref. 12: p. 19]. All of the functions and formulas developed above play an important role in generalizing the ordinary orthogonal polynomial classes.

B. BASIC EXTENSIONS OF ORTHOGONAL POLYNOMIALS

In this section, we present the $q-$analogue(s) of selected classes from earlier chapters. By using formulas such as those presented in the previous section together with methods based on those outlined in the preceding chapters, it is possible to derive the recurrence relations, difference equations, and Rodrigues' formulas as well as many other identities satisfied by these $q-$versions. As mentioned in the previous section, the $q-$extension of a function is not necessarily unique; however the last two classes listed are especially important.

1. Continuous $q-$Hermite Polynomials

a. *Definition*

$$
H_n(x\,|\,q)=\sum_{k=0}^{n}\frac{(q\,;q)_n}{(q\,;q)_k\,(q\,;q)_{n-k}}\,e^{i(n-2k)\theta}
$$

where $x=\cos\theta$.

b. Orthogonality Relation

$$\int_0^\pi H_m(\cos\theta \mid q)\, H_n(\cos\theta \mid q)\, \left|(e^{2i\theta}; q)_\infty\right|^2 d\theta = \frac{2\pi\,(q;q)_n}{(q;q)_\infty}\,\delta_{m,n}$$

where $x = \cos\theta$. [Ref. 12: p. 188]

2. Discrete q −Hermite Polynomials

a. Definition

$$H_n(x;q) = \sum_{k=0}^{[n/2]} \frac{(q;q)_n}{(q^2;q^2)_k\,(q;q)_{n-2k}}\,(-1)^k\, q^{k(k-1)}\, x^{n-2k}$$

b. Orthogonality Relation

$$\int_{-1}^{1} H_m(x;q)\, H_n(x;q)\, d\psi(x) = q^{\binom{n}{2}}\,(q;q)_n\,\delta_{m,n}$$

where $\psi(x)$ is a step function with jumps

$$\frac{|\,x\,|}{2}\;\frac{(x^2q^2;q^2)_\infty\,(q;q^2)_\infty}{(q^2;q^2)_\infty}$$

at the points $x = \pm q^j$, $j = 0, 1, 2, \ldots$. [Ref. 12: p. 193]

3. q −Laguerre Polynomials

a. Definition

For $\alpha > -1$,

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\;{}_1\phi_1\left[\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix};\, q,\, -xq^{n+\alpha+1}\right].$$

b. Continuous Orthogonality

$$\int_0^\infty L_m^{(\alpha)}(x;q)\, L_n^{(\alpha)}(x;q)\, \frac{x^\alpha\, dx}{(-(1-q)x;q)_\infty} = \frac{\Gamma(\alpha+1)\,\Gamma(-\alpha)\,(q^{\alpha+1};q)_n}{\Gamma_q(-\alpha)\,(q;q)_n\, q^n}\,\delta_{m,n}$$

c. *Discrete Orthogonality*

$$\sum_{k=-\infty}^{\infty} L_m^{(\alpha)}(cq^k ; q)\, L_n^{(\alpha)}(cq^k ; q)\ \frac{q^{k(\alpha+1)}}{\left(-c(1-q)q^k ; q\right)_\infty} = A\ \frac{\left(q^{\alpha+1} ; q\right)_n}{(q ; q)_n\, q^n}\ \delta_{m,n}$$

where

$$A = \frac{(q ; q)_\infty \left(-c(1-q)q^{\alpha+1} ; q\right)_\infty \left(-(1-q)/(cq^\alpha) ; q\right)_\infty}{\left(q^{\alpha+1} ; q\right)_\infty \left(-c(1-q) ; q\right)_\infty \left(-\frac{q}{c}(1-q) ; q\right)_\infty}$$

[Ref. 12: pp. 194-195]

4. Little q –Jacobi Polynomials

a. *Definition*

$$p_n(x ; a, b ; q) = {}_2\phi_1 \left[\begin{matrix} q^{-n},\ abq^{n+1} \\ aq \end{matrix} ; q,\ xq\right]$$

b. *Orthogonality Relation*

$$\sum_{x=0}^{\infty} p_m(q^x ; a, b ; q)\, p_n(q^x ; a, b ; q)\ \frac{(bq ; q)_x}{(q ; q)_x}\ (aq)^x$$
$$= \frac{(q ; q)_n (1-abq)\, (bq ; q)_n \left(abq^2 ; q\right)_\infty}{(abq ; q)_n \left(1-abq^{2n+1}\right) (aq ; q)_n\, (aq ; q)_\infty}\ (aq)^n\, \delta_{m,n}$$

where $0 < q$, $aq < 1$ [Ref. 12: p. 166]

5. Big q –Jacobi Polynomials

a. *Definition*

$$P_n(x ; a, b, c ; q) = {}_3\phi_2 \left[\begin{matrix} q^{-n},\ abq^{n+1},\ x \\ aq,\ cq \end{matrix} ; q,\ q\right]$$

b. Orthogonality Relation

$$\int_{cq}^{aq} P_m(x\,;a,b,c\,;q)\,P_n(x\,;a,b,c\,;q)\,\frac{(x/a\,;q)_\infty\,(x/c\,;q)_\infty}{(x\,;q)_\infty\,(bx/c\,;q)_\infty}\,d_qx$$
$$= M\,\frac{(q\,;q)_n\,(1-abq)\,(bq\,;q)_n\,(abq/c\,;q)_n}{(abq\,;q)_n\,\left(1-abq^{2n+1}\right)(aq\,;q)_n\,(cq\,;q)_n}\,(-ac)^{-n}\,q^{\binom{n}{2}}\,\delta_{m,n}$$

where

$$M=\int_{cq}^{aq}\frac{(x/a\,;q)_\infty\,(x/c\,;q)_\infty}{(x\,;q)_\infty\,(bx/c\,;q)_\infty}\,d_qx$$
$$=\frac{aq\,(1-q)\,(q\,;q)_\infty\,(c/a\,;q)_\infty\,(aq/c\,;q)_\infty\,\left(abq^2\,;q\right)_\infty}{(aq\,;q)_\infty\,(bq\,;q)_\infty\,(cq\,;q)_\infty\,(abq/c\,;q)_\infty}$$

[Ref. 12: pp. 167-168]

6. q –Krawtchouk Polynomials

a. Definition

$$K_n(x\,;a,N\,;q)={}_3\phi_2\left[\begin{matrix}q^{-n},\ x,\ -a^{-1}q^n\\ q^{-N},\ 0\end{matrix}\,;q,\ q\right]$$

b. Orthogonality Relation

$$\sum_{x=0}^{N} K_m(q^{-x}\,;a,N\,;q)\,K_n(q^{-x}\,;a,N\,;q)\,\frac{(q^{-N}\,;q)_x}{(q\,;q)_x}\,(-a)^x$$
$$=\left(-qa^{-1}\,;q\right)_N a^N\,q^{-\binom{N+1}{2}}\,\frac{(q\,;q)_n\left(1+a^{-1}\right)\left(-a^{-1}q^{N+1}\,;q\right)_n}{\left(-a^{-1}\,;q\right)_n\left(1+a^{-1}q^{2n}\right)\left(q^{-N}\,;q\right)_n}$$
$$\times\left(-aq^{N+1}\right)^{-n}q^{n(n+1)}\,\delta_{m,n}$$

[Ref. 12: p. 185]

7. q –Hahn Polynomials

a. Definition

$$Q_n(x\,;a,b,N\,;q)={}_3\phi_2\left[\begin{matrix}q^{-n},\ abq^{n+1},\ q^{-x}\\ aq,\ q^{-N}\end{matrix}\,;q,\ q\right]$$

b. Orthogonality Relation

$$\sum_{x=0}^{N} Q_m(x\,;a,b,N\,;q)\,Q_n(x\,;a,b,N\,;q)\;\frac{(aq\,;q)_x\,(bq\,;q)_{N-x}}{(q\,;q)_x\,(q\,;q)_{N-x}}\,(aq)^{-x}$$

$$=\frac{\left(abq^2\,;q\right)_N (aq)^{-N}}{(q\,;q)_N}\;\frac{(q\,;q)_n\,(1-abq)\,(bq\,;q)_n\left(abq^{N+2}\,;q\right)_n}{(abq\,;q)_n\left(1-abq^{2n+1}\right)(aq\,;q)_n\left(q^{-N}\,;q\right)_n}$$

$$\times\,(-aq)^n\,q^{\binom{n}{2}-Nn}\,\delta_{m,n}\,,\quad m,\ n=0,1,\dots,N$$

[Ref. 12: p. 165]

8. **Dual q —Hahn Polynomials**

a. Definition

$$R_n(\,\mu(x)\,;b,c,N\,;q)={}_3\phi_2\left[\begin{matrix}q^{-n},\ q^{-x},\ cq^{x-N}\\ q^{-N},\ bcq\end{matrix}\;;q,\ q\right]$$

where $\mu(x)=q^{-x}+cq^{x-N}$

b. Orthogonality Relation

$$\sum_{x=0}^{N} R_m(\mu(x)\,;b,c,N\,;q)\,R_n(\mu(x)\,;b,c,N\,;q)$$

$$\times\,\frac{\left(cq^{-N}\,;q\right)_x\left(1-cq^{2x-N}\right)(bcq\,;q)_x\left(q^{-N}\,;q\right)_x}{(q\,;q)_x\left(1-cq^{-N}\right)\left(b^{-1}q^{-N}\,;q\right)_x\,(\,'\,;q)_x}\;q^{Nx-\binom{x}{2}}\,(-bcq)^{-x}$$

$$=\frac{(1/c\,;q)_N}{(bq\,;q)_N}\;\frac{(q\,;q)_n\,(bq\,;q)_n}{(bcq\,;q)_n\left(q^{-N}\,;q\right)_n}\;\left(cq^{-N}\right)^n\delta_{m,n}\,,\quad m,\ n=0,1,\dots,N$$

[Ref. 12: p. 166]

9. **q —Racah Polynomials**

a. Definition

(8) $$p_n(\mu(x)\,;a,b,c,d\,;q)={}_4\phi_3\left[\begin{matrix}q^{-n},\ abq^{n+1},\ q^{-x},\ cdq^{x+1}\\ aq,\ bdq,\ cq\end{matrix}\;;q,\ q\right]$$

where $\mu(x)=q^{-x}+cdq^{x+1}$

b. Orthogonality Relation

When aq, bdq, or $cq = q^{-N}$ for a positive integer N, then

(9) $$\sum_{x=0}^{N} p_m(\mu(x)\,;a,b,c,d\,;q)\, p_n(\mu(x)\,;a,b,c,d\,;q)\, w(x\,;a,b,c,d\,;q) = h_n\,\delta_{m,n}$$

where

(10) $$w(x\,;a,b,c,d\,;q) = \frac{(cdq\,;q)_x\,(aq\,;q)_x\,(bdq\,;q)_x\,(cq\,;q)_x}{(q\,;q)_x\,(a^{-1}cdq\,;q)_x\,(b^{-1}cq\,;q)_x\,(dq\,;q)_x\,(abq)^x}\;\frac{1-cdq^{2x+1}}{1-cdq}$$

and

(11) $$h_n = \frac{(q\,;q)_n\,(1-abq)\,(bq\,;q)_n\,(ad^{-1}q\,;q)_n\,(abc^{-1}q\,;q)_n\,(cdq)^n}{(abq\,;q)_n\,(1-abq^{2n+1})\,(aq\,;q)_n\,(bdq\,;q)_n\,(cq\,;q)_n} \times \frac{(cdq^2\,;q)_\infty\,(a^{-1}b^{-1}c\,;q)_\infty\,(a^{-1}d\,;q)_\infty\,(b^{-1}\,;q)_\infty}{(a^{-1}cdq\,;q)_\infty\,(b^{-1}cq\,;q)_\infty\,(dq\,;q)_\infty\,(a^{-1}b^{-1}q^{-1}\,;q)_\infty}$$

[Ref. 20: p. 1014]. When aq, bdq, or cq is equal to q^{-N}, the infinite products in (11) reduce to finite products. Hence the orthogonality relation (9) is valid for all q provided no zeros are introduced into denominator terms [Ref. 18: p. 4].

10. Askey-Wilson Polynomials

a. Definition

(12) $$p_n(x\,;a,b,c,d\mid q) = a^{-n}\,(ab\,;q)_n\,(ac\,;q)_n\,(ad\,;q)_n \times {}_4\phi_3\left[\begin{matrix} q^{-n},\ abcdq^{n-1},\ ae^{i\theta},\ ae^{-i\theta} \\ ab,\ ac,\ ad \end{matrix}\,;q,\ q\right]$$

where $x = \cos\theta$ [Ref. 18: p. 3]

b. Orthogonality Relation

For $-1 < a,\ b,\ c,\ d,\ q < 1$,

$$\frac{1}{2\pi}\int_{-1}^{1} p_m(x\,;a,b,c,d\mid q)\, p_n(x\,;a,b,c,d\mid q)\,\frac{w(x\,;a,b,c,d\mid q)\,dx}{\sqrt{1-x^2}} = h_n\,\delta_{m,n}$$

where

$$w(x\,;a,b,c,d \mid q) = \frac{\prod_{k=0}^{\infty}\left(1 - 2(2x^2-1)q^k + q^{2k}\right)}{h(x,a)\,h(x,b)\,h(x,c)\,h(x,d)}$$

with

$$h(x,a) = \prod_{k=0}^{\infty}\left(1 - 2axq^k + a^2q^{2k}\right)$$

and

$$h_n = \frac{\left(abcdq^{2n}\,;q\right)_\infty \left(abcdq^{n-1}\,;q\right)_n \left(q^{n+1}\,;q\right)_\infty^{-1} \left(abq^n\,;q\right)_\infty^{-1}}{\left(acq^n\,;q\right)_\infty \left(adq^n\,;q\right)_\infty \left(bcq^n\,;q\right)_\infty \left(bdq^n\,;q\right)_\infty \left(cdq^n\,;q\right)_\infty}\,.$$

[Ref. 18: pp. 11-14]

We can establish a formal connection between these last two classes. If the q −Racah polynomials are written

$$_4\phi_3\left[\begin{matrix} q^{-n},\; a'b'q^{n+1},\; q^{-x},\; c'd'q^{x+1} \\ a'q,\; b'd'q,\; c'q \end{matrix}\;;q,\; q\right]$$

and the following parameter changes are made:

$$a' = \frac{ac}{q} \quad , \; b' = \frac{bd}{q} \quad , \; c' = \frac{ab}{q} \quad , \; d' = \frac{a}{b}\,.$$

(or likewise using any permutation of $\{\,b,c,d\,\}$ assuming at least one of these is non-zero), as well as the change of variable $q^{-x} = ae^{i\theta}$, then we obtain

$$_4\phi_3\left[\begin{matrix} q^{-n},\; abcdq^{n-1},\; ae^{i\theta},\; ae^{-i\theta} \\ ab,\; ac,\; ad \end{matrix}\;;q,\; q\right]$$

with

$$\begin{aligned}\mu(x) &= q^{-x} + q^{x+1}c'd' \\ &= ae^{i\theta} + ae^{-i\theta} = 2a\cos\theta\end{aligned}$$

That is, the q −Racah and Askey-Wilson polynomials are virtually the same, differing only in their parameters, normalization, and variable.

Because the q −Racah and Askey-Wilson polynomials are essentially the same, the names are often used interchangeably in the literature. We have presented these polynomials as distinct classes in order to emphasize the continuous and discrete natures as expressed in the orthogonality relations.

As stated at the beginning of Chapter IV, the "classical orthogonal polynomials" are defined to be those which are special or limiting cases of the Askey-Wilson (21) or q −Racah polynomials (18) [Ref. 31: p. 57]. We can now make this statement a bit more precise. Letting

$$a = q^{\alpha} \quad , \quad b = q^{\beta} \quad , \quad c = q^{\gamma} \quad , \quad d = q^{\delta}$$

in (9) and (10) and taking the limit as $q \to 1$, the ordinary Racah polynomials $R_n(\lambda(x); \alpha, \beta, \gamma, \delta)$ and their weight function as defined in Formulas (30) and (32), Chapter IV, Section C.3 are retrieved. A similar limiting process will recover any ordinary orthogonal polynomial class from its q −extension. Moreover, any q −extension previously discussed is a special or limiting case of the q −Racah. For example, letting $cq = q^{-N}$ and $d = 0$ produces the q −Hahn, etc. In this way, we see that the self-dual q −Racah polynomials encompass all the previous classes. They also satisfy three-term recurrence relations, second order difference equations, and Rodrigues' formulas with respect to q −divided difference operators. The interested reader is referred to [Refs. 18, 12, 20] for details about these very rich classes.

C. CONCLUDING REMARKS

In Chapter V, we presented a few of the traditional applications in which orthogonal polynomials have played an important part. More recently, an enriching interplay has developed between the theory of orthogonal polynomials and other mathematical and mathematically-related areas.

Efficient computational methods have been devised for determining the many useful quantities associated with orthogonal polynomials (such as their zeros, recurrence coefficients, etc.) [Ref. 32: pp. 181-216]. Various classes of orthogonal polynomials have also played a role in digital signal processing [Ref. 32: pp. 115-133], quantum mechanics [Ref. 32: pp. 217-228], and birth/death processes [Ref. 32: pp. 229-255]. Advances in the field of combinatorics and graph theory have allowed new geometric interpretations of

orthogonal polynomial identities, some of which have very important consequences for "association schemes" and the designs of codes [Refs. 33, 16, 34, 32 : pp. 25-53, 35].

Physicists have introduced various versions of "diagrammatic methods": ways of understanding orthogonal polynomials through 3-j and 6-j symbols, their generalizations, and accompanying identities by formally associating them with pictorial schematics representing forces of physical systems that conserve angular momentum [Ref. 36]. Powerful new techniques involving "quantum groups" have been used to generate new identities for some classes [Ref. 32: pp. 257-292]. Further investigation into the electrostatics problem discussed in the text has led to the formation of the famed "Selberg beta integral" and its generalizations. It has yielded to analysis via the study of the "root systems" of Lie algebras, and has found applications ranging from statistical mechanics to computer algorithm complexity [Refs. 30: pp. 48-52, 32 : pp. 311-318, 37].

Finally, research into the general structure of q-series has led to many surprising connections, and is intimately related to the many remarkable and powerful number-theoretic formulas discovered by S. Ramanujan, the famed Indian mathematical prodigy [Refs. 30: pp. 87-93, 38: pp. 55-66, 39].

This utility and promise of future applications provide ample motivation and justification for continued study of the intrinsic structure of orthogonal polynomials.

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